A few words on connected diagrams, size-extensivity, and all that...

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Structure behind this mess?

$$\mathbf{E_n^{(1)}} = V_{nn}$$

$$\mathbf{E_n^{(2)}} = \frac{|V_{nk_2}|^2}{E_{nk_2}}$$

$$\mathbf{E_n^{(3)}} = \frac{V_{nk_3}V_{k_3k_2}V_{k_2n}}{E_{nk_2}E_{nk_3}} - V_{nn}\frac{|V_{nk_3}|^2}{E_{nk_3}^2}$$

$$\mathbf{E_n^{(4)}} = \frac{V_{nk_4}V_{k_4k_3}V_{k_3k_2}V_{k_2n}}{E_{nk_2}E_{nk_3}E_{nk_4}} - \frac{|V_{nk_4}|^2}{E_{nk_4}^2}\frac{|V_{nk_2}|^2}{E_{nk_2}} - V_{nn}\frac{V_{nk_4}V_{k_4k_3}V_{k_3n}}{E_{nk_3}^2E_{nk_4}} - V_{nn}\frac{V_{nk_4}V_{k_4k_2}V_{k_2n}}{E_{nk_2}E_{nk_4}^2} + V_{nn}^2\frac{|V_{nk_4}|^2}{E_{nk_4}^3}$$

$$= \frac{V_{nk_4} V_{k_4 k_3} V_{k_3 k_2} V_{k_2 n}}{E_{nk_2} E_{nk_3} E_{nk_4}} - E_n^{(2)} \frac{|V_{nk_4}|^2}{E_{nk_4}^2} - 2 V_{nn} \frac{V_{nk_4} V_{k_4 k_3} V_{k_3 n}}{E_{nk_3}^2 E_{nk_4}} + V_{nn}^2 \frac{|V_{nk_4}|^2}{E_{nk_4}^3}$$

$$\mathbf{E_n^{(5)}} = \frac{V_{nk_5} V_{k_5 k_4} V_{k_4 k_3} V_{k_3 k_2} V_{k_2 n}}{E_{nk_2} E_{nk_3} E_{nk_4} E_{nk_5}} - \frac{V_{nk_5} V_{k_5 k_4} V_{k_4 n}}{E_{nk_4}^2 E_{nk_5}} \frac{|V_{nk_2}|^2}{E_{nk_2}}$$

$$- \frac{V_{nk_5} V_{k_5 k_2} V_{k_2 n}}{E_{nk_2} E_{nk_5}^2} \frac{|V_{nk_2}|^2}{E_{nk_2}} - \frac{|V_{nk_5}|^2}{E_{nk_5}^2} \frac{V_{nk_3} V_{k_3 k_2} V_{k_2 n}}{E_{nk_2} E_{nk_3}}$$

$$-V_{nn} \frac{V_{nk_5} V_{k_5 k_4} V_{k_4 k_3} V_{k_3 n}}{E_{nk_3}^2 E_{nk_4} E_{nk_5}} - V_{nn} \frac{V_{nk_5} V_{k_5 k_4} V_{k_4 k_2} V_{k_2 n}}{E_{nk_2} E_{nk_4}^2 E_{nk_5}} - V_{nn} \frac{V_{nk_5} V_{k_5 k_3} V_{k_3 k_2} V_{k_2 n}}{E_{nk_2} E_{nk_3} E_{nk_5}^2} + V_{nn} \frac{|V_{nk_5}|^2}{E_{nk_5}^2} \frac{|V_{nk_3}|^2}{E_{nk_3}^2} +$$

$$2 V_{nn} \frac{|V_{nk_5}|^2}{E_{nk_5}^3} \frac{|V_{nk_2}|^2}{E_{nk_2}} + V_{nn}^2 \frac{V_{nk_5} V_{k_5 k_4} V_{k_4 n}}{E_{nk_4}^3 E_{nk_5}} + V_{nn}^2 \frac{V_{nk_5} V_{k_5 k_3} V_{k_3 n}}{E_{nk_3}^2 E_{nk_5}^2} + V_{nn}^2 \frac{V_{nk_5} V_{k_5 k_2} V_{k_2 n}}{E_{nk_2} E_{nk_5}^3} - V_{nn}^3 \frac{|V_{nk_5}|^2}{E_{nk_5}^4}$$

$$\mathbf{E_n^{(5)}} = \frac{V_{nk_5}V_{k_5k_4}V_{k_4k_3}V_{k_3k_2}V_{k_2n}}{E_{nk_2}E_{nk_3}E_{nk_4}E_{nk_5}} - 2E_n^{(2)}\frac{V_{nk_5}V_{k_5k_4}V_{k_4n}}{E_{nk_4}^2E_{nk_5}} - \frac{|V_{nk_5}|^2}{E_{nk_5}^2}\frac{V_{nk_3}V_{k_3k_2}V_{k_2n}}{E_{nk_2}E_{nk_3}}$$

$$\text{-2V}_{nn}\left(\frac{V_{nk_5}V_{k_5k_4}V_{k_4k_3}V_{k_3n}}{E_{nk_3}^2E_{nk_4}E_{nk_5}} - \frac{V_{nk_5}V_{k_5k_4}V_{k_4k_2}V_{k_2n}}{E_{nk_2}E_{nk_4}^2E_{nk_5}} + \frac{|V_{nk_5}|^2}{E_{nk_5}^2}\frac{|V_{nk_3}|^2}{E_{nk_3}^2} + 2E_n^{(2)}\frac{|V_{nk_5}|^2}{E_{nk_5}^3}\right)$$

$$+\text{V}_{nn}^2\left(2\frac{V_{nk_5}V_{k_5k_4}V_{k_4n}}{E_{nk_4}^3E_{nk_5}} + \frac{V_{nk_5}V_{k_5k_3}V_{k_3n}}{E_{nk_3}^2E_{nk_5}^2}\right) - V_{nn}^3\frac{|V_{nk_5}|^2}{E_{nk_5}^4}$$

In virtually all fundamental theories of interacting particles and fields, physical properties can be expressed using

- The definition of a configuration space consisting of all possible configurations C for the system
- The definition of a "probability" for a given state

$$\boxed{\pi(\mathcal{C}) \sim e^{-\int \mathcal{H}/\mathcal{L}}} \tag{1}$$

where $\mathcal{H}$ = Hamiltonian ,$\mathcal{L}$ = Lagrangian.

Properties expressed as

$$\bar{Q} = \langle Q \rangle_{\pi}$$

- Classical statistical physics: $\mathcal{C} = (x, v)$ $\pi(\mathcal{C}) \sim e^{-\beta E(\mathcal{C})}$
- Quantum statistical physics: $\mathcal{C} = \text{path} = x(t)$ $\pi(\mathcal{C}) \sim e^{-\int_0^\beta dt \mathcal{H}(x(t))}$
- Quantum mechanics in real time: $\mathcal{C} = \text{path} = x(t)$ $\pi(\mathcal{C}) \sim e^{-\int_0^\beta dt \mathcal{L}(x(t))}$
- Quantum mechanics in imaginary time ($t \to it$, QMC):

$\mathcal{C} = \text{path} = x(t)$ $\pi(\mathcal{C}) \sim e^{-\int_0^\beta dt \mathcal{H}(x(t))}$

- Quantum Field Theory: $\mathcal{C} = \text{field} = \phi(x, t)$ $\pi(\mathcal{C}) \sim e^{-\int dx dt \mathcal{H}/\mathcal{L}(\phi)}$

Let us call Z the normalization of the probability

$$\pi(\mathcal{C}) = \frac{e^{-\int \mathcal{H}/\mathcal{L}}}{Z}$$

$$Z = \int_{\text{all } C} e^{-\int \mathcal{H}/\mathcal{L}}$$

In all these theories the ground-state energy can be written as the logarithm of the normalization

$$\boxed{E_0 \sim \ln Z = \ln \langle e^{-\int \mathcal{H}/\mathcal{L}} \rangle} \tag{2}$$

that **the logarithm of an average of an exponential function**

This is this property which is **at the root of the ”connected structure” of all these theories**

Let us see how it works in quantum mechanics

Let us define some reference Hamiltonian H_0 so that

$$H_0\Phi_i^{(0)} = E_i^{(0)}\Phi_i^{(0)}$$

and

$$H = H_0 + \lambda V$$

with

$$H\Phi_i = E_i\Phi_i$$

All exact eigenstates are taken normalized here. The ground-state energy can be written as

$$E_0 = \lim_{t\to+\infty} -\frac{1}{t}\ln\langle\Phi_0^{(0)}|e^{-tH}\Phi_0^{(0)}\rangle$$

This property results from

$$e^{-tH}|\Phi_0^{(0)}\rangle = \sum_i e^{-tE_i}\langle\Phi_i|\Phi_0^{(0)}\rangle|\Phi_i\rangle$$

with $E_0 \leq E_1 \leq E_2 \leq \ldots$ and then

$$\langle\Phi_0^{(0)}e^{-tH}|\Phi_0^{(0)}\rangle \sim_{t\to+\infty} e^{-tE_0}\langle\Phi_0|\Phi_0^{(0)}\rangle^2 + O(e^{-t(E_1-E_0)})$$

Path integral formalism of quantum mechanics

$$\langle\Phi_0^{(0)}|e^{-tH}\Phi_0^{(0)}\rangle = \langle\Phi_0^{(0)}|e^{-\tau(H_0+\lambda V)}e^{-\tau(H_0+\lambda V)}...e^{-\tau(H+\lambda V)}|\Phi_0^{(0)}\rangle$$

where $\tau = t/N$. Now, we use the following decomposition

$$\langle x|e^{-\tau(H_0+\lambda V)}|x'\rangle = \langle x|e^{-\tau H_0}|x'\rangle e^{-\tau\lambda V(x)}\delta(x-x')$$

which is **exact** in the $\tau = 0$-limit. So that

$$\langle\Phi_0^{(0)}e^{-tH}|\Phi_0^{(0)}\rangle = lim_{N\to+\infty}$$

$$\int dx_0...dx_{N-1}\langle\Phi_0^{(0)}|e^{-\tau H_0}|x_1\rangle\langle x_1|e^{-\tau H_0}|x_2\rangle...\langle x_{N-1}|e^{-\tau H_0}|\Phi_0^{(0)}\rangle e^{-\lambda\tau\sum_{i=1}^{N}V(x_i)}$$

We define the density of probability

$$\pi(x_1,...x_{N-1}) = \langle\Phi_0^{(0)}|e^{-\tau(H_0-E_0^{(0)})}|x_1\rangle\prod_{i=1}^{N-2}\langle x_i|e^{-\tau(H_0-E_0^{(0)})}|x_{i+1}\rangle\langle x_{N-1}|e^{-\tau(H_0-E_0^{(0)})}|\Phi_0^{(0)}\rangle$$

We have

$$\pi \geq 0$$

and

$$\int dx_1...dx_{N-1}\pi(x_1,...x_{N-1}) = \langle\Phi_0^{(0)}|e^{-t(H_0-E_0^{(0)})}|\Phi_0^{(0)}\rangle = 1$$

So

$$E_0 = \lim_{t \to +\infty} -\frac{1}{t} \ln \left[e^{-tE_0^{(0)}} \left\langle e^{-\lambda \int_0^t V[x(t)]} \right\rangle_\pi \right]$$

and we have the **exact equality** expressing the energy as the log of the average of an exponential

$$\boxed{E_0 = E_0^{(0)} - \lim_{t \to +\infty} -\frac{1}{t} \ln \left\langle e^{-\lambda \int_0^t V[x(t)]} \right\rangle_\pi} \tag{3}$$

Let us decompose the exponential

$$E_0 \sim \ln \left\langle \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \int_0^t dt_1 \ldots \int_0^t dt_n V[x(t_1)] \ldots V[x(t_n)] \right\rangle_\pi$$

$$\sim \ln \left[1 - \frac{\lambda}{1!} \int_0^t d1 \langle V_1 \rangle + \frac{\lambda^2}{2!} \int_0^t \int_0^t d1 d2 \langle V_1 V_2 \rangle + \ldots\right]$$

$$\sim \ln \left[1 - \lambda \int_0^t d1 \langle V_1 \rangle + \lambda^2 \int_0^t \int_{t_1}^t d1 d2 \langle V_1 V_2 \rangle + \lambda^3 \int_0^t \int_{t_1}^t \int_{t_2}^t d1 d2 d3 \langle V_1 V_2 V_3 \rangle \ldots\right]$$

Denoting

$$R(t) = e^{-t(H_0 - E_0^{(0)})}$$

$$\int_0^t d1 \langle V_1 \rangle = \int_0^t d1 \langle \Phi_0^{(0)} | R(t_1) V R(t - t_1) | \Phi_0^{(0)} \rangle = t \langle \Phi_0^{(0)} | V | \Phi_0^{(0)} \rangle$$

$$\int_0^t \int_{t_1}^t d1 d2 \langle V_1 V_2 \rangle = \int_0^t \int_{t_1}^t d1 d2 \langle \Phi_0^{(0)} | R(t_1) V R(t_2 - t_1) V R(t - t_2) | \Phi_0^{(0)} \rangle$$

$$= t \int_0^t du_1 \Phi_0^{(0)} | V R(u_1) V | \Phi_0^{(0)} \rangle$$

$$\int_0^t \int_{t_1}^t \int_{t_2}^t du_1 du_2 \langle V_1 V_2 V_3 \rangle = t \int_0^t \int_{u_2}^t \Phi_0^{(0)} | V R(u_1) V R(u_2) V | \Phi_0^{(0)} \rangle$$

etc.

We have

$$\ln(1+x) = 1 - x + x^2/2 + \dots.$$

Then

$$E_0 = E_0^{(0)} + \lambda\langle V_1\rangle - \lambda^2 \int_0^\infty \langle (V_1 - \langle V\rangle)(V_2 - \langle V\rangle)\rangle + \dots$$

- First order

$$E_1 = \langle V_1\rangle = \langle \Phi_0^{(0)}|V|\Phi_0^{(0)}\rangle$$

- Second-order

$$E_2 = -\int_0^\infty \langle (V_1 - \langle V\rangle)(V_2 - \langle V\rangle)\rangle = -\int_0^\infty du \langle \Phi_0^{(0)}|(V - \langle V\rangle)e^{-u(H_0 - E_0^{(0)})}(V - \langle V\rangle)|\Phi_0^{(0)}\rangle$$

$$= -\sum_{i=0} \langle \Phi_0^{(0)}|(V - \langle V\rangle)|\Phi_i^{(0)}\rangle^2 \int_0^\infty du e^{-u(E_i^{(0)} - E_0^{(0)})} = -\sum_{\mathbf{i\neq 0}} \langle \Phi_0^{(0)}|V\Phi_i^{(0)}\rangle^2 \int_0^\infty du e^{-u(E_i^{(0)} - E_0^{(0)})}$$

$$= -\sum_{i\neq 0} \frac{\Phi_0^{(0)}|V|\Phi_i^{(0)}\rangle^2}{E_i^{(0)} - E_0^{(0)}}$$

Let X be random variable with probability $\pi(x)$. The moments are defined as

$$M_n = \int \pi(x) x^n = \langle x^n \rangle$$

Generating function for the moments

$$G(\lambda) = \langle e^{\lambda x} \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\partial^n \langle e^{\lambda X} \rangle}{\partial \lambda^n}(0) \lambda^n = \sum_{n=0}^{\infty} \frac{M_n}{n!} \lambda^n$$

Here

$$E_0(\lambda) \sim \ln\langle e^{\lambda X}\rangle_\pi$$

where the random variable X is

$$X = -\int_0^t V[x(t)]$$

Because of the logarithm **the moments are not the appropriate quantities.** Indeed, we would like instead

$$\ln\langle e^{\lambda x}\rangle_\pi = \sum_{n=0}^{\infty} \frac{C_n}{n!}\lambda^n$$

that is

$$\langle e^{\lambda x}\rangle = \sum_{n=0}^{\infty} \frac{M_n}{n!}\lambda^n = \exp\left\{\sum_{n=0}^{\infty} \frac{C_n}{n!}\lambda^n\right\}$$

The C_n are the **cumulants.**

Easy to calculate by expanding the exponential

$$
\begin{aligned}
C_1 &= M_1 \\
C_2 &= M_2 - {M_1}^2 \\
C_3 &= M_3 - 3M_2M_1 + 2{M_1}^3 \\
C_4 &= M_4 - 4M_3M_1 - 3{M_2}^2 + 12M_2{M_1}^2 - 6{M_1}^4 \\
C_5 &= M_5 - 5M_4M_1 - 10M_3M_2 + 20M_3{M_1}^2 + 30{M_2}^2M_1 - 60M_2{M_1}^3 + 24{M_1}^5 \\
C_6 &= M_6 - 6M_5M_1 - 15M_4M_2 + 30M_4{M_1}^2 - 10{M_3}^2 + 120M_3M_2M_1 \\
&\quad - 120M_3{M_1}^3 + 30{M_2}^3 - 270{M_2}^2{M_1}^2 + 360M_2{M_1}^4 - 120{M_1}^6
\end{aligned}
$$

$$\boxed{E_0 = E_0^{(0)} - \lim_{t \to +\infty} \frac{1}{t} \ln \langle e^{-\lambda \int_0^t V[x(t)]} \rangle_\pi} \tag{4}$$

If the system decomposes into two independent parts

$$V = V_A + V_B$$

$$\pi = \pi_A \pi_B$$

$$E_0 = E_0^{(0)} - \lim_{t \to +\infty} \frac{1}{t} \ln \left[\langle e^{-\lambda \int_0^t V_A[x_A(t)]} \rangle_{\pi_A} \langle e^{-\lambda \int_0^t V_B[x_B(t)]} \rangle_{\pi_B} \right]$$

$$E_0 = E_0^A + E_0^B$$

But we have also

$$E_n = -\lim_{t\to\infty} \frac{1}{t} \frac{\partial^n ln\langle e^{-\lambda \int V}\rangle}{\partial \lambda^n}(0)$$

$$= -\lim_{t\to\infty} \frac{1}{t} \left[\frac{\partial^n ln\langle e^{-\lambda \int V_A}\rangle}{\partial \lambda^n}(0) + \frac{\partial^n ln\langle e^{-\lambda \int V_B}\rangle}{\partial \lambda^n}(0) \right]$$

which gives

$$E_n = E_n^A + E_n^B$$

The energy is size-extensive **order by order** in perturbation

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