Transitional regions of finite Fermi systems and quantum chaos

W. D. Heiss and A. L. Sannino
Centre for Nonlinear Studies and Department of Physics, University of the Witwatersrand, P.O. Wits 2050, Johannesburg, South Africa
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It is argued that phase transitions in finite systems are likely to exhibit level statistics ascribed to quantum chaos. Apparent counterexamples become chaotic under generic perturbation. The mathematical mechanism for this to occur lies in the abundance of exceptional points in the transitional region. Exceptional points are singularities in the complex λ plane of the spectrum $E_n(\lambda)$, where λ is the parameter that effects the transition. Examples are given for illustration. It is shown that the quality of the mean field is unaffected outside the transitional region whether or not the transitional region is chaotic. Related properties of the eigenvectors are also discussed.

I. INTRODUCTION

Classical systems that are nonintegrable show generically chaotic behavior. The quantum energy levels of such systems, if they can be straightforwardly quantized, exhibit particular patterns that are denoted as quantum chaos.[1] If a classical system is integrable it is usually due to symmetries of the Hamilton function giving rise to first integrals. When the symmetry is perturbed chaotic motion can set in. The question arises whether a corresponding pattern prevails in quantum mechanics.

The occurrence of energy-level statistics which are ascribed to quantum chaos[2] has been reported in a number of cases relating to symmetry breaking. A particularly well-studied situation is the problem of the hydrogen atom in a strong magnetic field.[3,4] There the transition from the regime of the Coulomb potential to that of a two-dimensional harmonic oscillator causes the energy levels to obey essentially the Gaussian orthogonal ensemble (GOE) statistics[2] for an intermediate range of λ values. Since in this case the classical analogy is known to be chaotic, the corresponding quantum-mechanical features are expected. However, in quantum-mechanical problems where the classical analogy is not easily at hand, the question arises whether GOE statistics of the energy levels, if they do occur in the transitional region, are a generic pattern or just flukes of some particular examples. Situations of particular interest are phase transitions of finite systems where a transition from one symmetry of the Hamiltonian to another is usually encountered.[5,6]

In formal language, the spectrum $E_n(\lambda)$ of the Hamiltonian $H_0+\lambda H_1$ is investigated. If the symmetries relating to H_0 and H_1 are incompatible or simply if the two operators do not commute, there is a transitional region of λ values where the level statistics may or may not display the characteristics of quantum chaos. The precise patterns, such as the energy ranges for which the GOE statistics are established at a particular λ value, vary strongly from case to case. A general understanding does not seem to be available. It is not even obvious whether or not the different symmetries relating to H_0 and H_1 are responsible for bringing about possible GOE statistics of the levels $E_n(\lambda)$ at some range of λ values.

In this paper we attempt to address this problem from a different viewpoint. It is accepted that quantum chaos is intimately connected to avoided level crossing. We recall that in a finite quantum system the phase transition is characterized by a transitional region where much level repulsion occurs. An example from nuclear physics of such a phase transition is the transition from a spherical to a deformed nucleus. When λ is sweeping over the transitional region the spectrum becomes soft, i.e., the gaps between the levels become narrow while they are wider for λ values smaller or larger than those of the transitional region. Here the signature of avoided level crossing is encountered on a large scale which is nicely demonstrated in the Lipkin model[7] revisited below. The crucial aspect of a phase transition is the dramatic change of the ground-state structure and that of the excited states which is in turn closely related to the occurrence of avoided level crossing. The mathematical mechanism which connects the two features, avoided level crossing and structural change of the states, lies in the exceptional points of the Hamiltonian.[8]

Exceptional points are singularities of the functions $E_n(\lambda)$. Generically they occur in the complex λ plane at complex conjugate values. They are associated with avoided level crossing.[9] That they are essential for the occurrence of a phase transition of finite systems has been discussed previously.[10,11] Whether or not the transitional region gives rise to level statistics relating to quantum chaos depend on the distribution of the exceptional points and the shape of the energy spectrum is of a similar nature to the connection between the poles of a scattering function and the resonance structure of the cross section. In the same way as the poles of the scattering function give rise to the shape of the cross section, the exceptional points bring about the shape of the spectrum, particularly the occurrence of avoided level crossing. It is this aspect which forms the major aim of this paper.

We restrict ourselves to finite-dimensional H_0 and H_1 as for this case the analytic structure of the functions

$E_n(\lambda)$ is well understood.[9] We do not believe that this is a crucial restriction since in all practical calculations the problems are eventually reduced to finite-dimensional matrices. We arrive at conclusions being guided by examples which we believe are relevant and significant although at this stage our conclusions are by and large empirical.

To render the paper self-contained we repeat some of the results of earlier works.[6,9–12] The following section is devoted to the discussion of exceptional points. This constitutes the more formal and hence the more rigorous part of the paper. In Sec. III we present results of particular examples. Aspects of mean fields and perturbation methods are discussed in Sec. IV while certain properties of the state vectors are discussed in Sec. V. The paper ends with a summary and a discussion.

II. EXCEPTIONAL POINTS

We assume that H_0 and H_1 are $N\times N$ real symmetric matrices. For the purpose of this paper we assume that the spectra of both operators are regular, i.e., without fluctuations.[2] We use the representation in which H_0 is diagonal. The spectrum $E_n(\lambda)$ of $H_\lambda=H_0+\lambda H_1$ is obtained from the roots of the secular equation

$$\det(E-H_\lambda)=0 \ . \tag{1}$$

Exceptional points are the points λ where two different roots coalesce.[8,13,14] This implies the additional simultaneous equation

$$\frac{d}{dE}\det(E-H_\lambda)=0 \ . \tag{2}$$

If we assume that H_λ is irreducible with respect to symmetries, the fulfillment of the two equations simultaneously is generically excluded for real λ as this would mean a genuine degeneracy of two levels. In turn, there are always complex solutions. In fact, the two equations are polynomials in E (and λ) of Nth and $(N-1)$th order, respectively. The variable E can be eliminated[15] to yield a polynomial of order $N(N-1)$ in λ.[16] Since the coefficients of the polynomial are real, the solutions, i.e., the exception points, come in complex conjugate pairs.

When taking the real and imaginary parts of Eqs. (1) and (2) separately the two equations above are four real equations for the four real solutions Re, Im, Reλ, and Imλ. A coalescence of two roots of Eq. (1) as enforced by Eq. (2) will therefore not allow an additional condition. We expect generically

$$\frac{d^2}{dE^2}\det(E-H_\lambda)\neq 0 \tag{3}$$

which excludes coalescence of three roots of Eq. (1). We also expect generically

$$\frac{d}{d\lambda}\det(E-H_\lambda)\neq 0 \tag{3'}$$

at the points where Eqs. (1) and (2) are obeyed. As a consequence, the polynomial

$$\prod_{n=1}^{N}[E-E_n(\lambda)]=\det(E-H_\lambda) \tag{4}$$

vanishes linearly in λ at the exceptional points. This implies that, if $E_n(\lambda)$ and $E_m(\lambda)$ coalesce at the complex value λ_c and assume there the (complex) value E_c, they must have the form

$$\begin{aligned} E_n(\lambda)&=E_c+\sum_{k=1}^{\infty}c_k(\sqrt{\lambda-\lambda_c})^k \ , \\ E_m(\lambda)&=E_c+\sum_{k=1}^{\infty}c_k(-\sqrt{\lambda-\lambda_c})^k \ . \end{aligned} \tag{5}$$

Only then can be the determinant have a simple zero in λ and a double zero E and be regular in E and λ. The expansions in Eq. (5) have a finite radius of convergence which is determined by the next-nearest singularity. From Eq. (5) we conclude that E_n and E_m are the value of one analytic function on two Riemann sheets which are connected by square-root branch points at λ_c and, because of the reality condition, its complex conjugate value $\bar{\lambda}_c$.

The $N(N-1)/2$ complex conjugate pairs of exceptional points connect analytically all possible pairs E_i and E_k, $i\neq k$ of eigenvalues. In other words, the N eigenvalues are the values of one analytic function (which is not explicitly available) on N different Riemann sheets. The natural numbering of the levels is given by the order $E_1\leq\cdots\leq E_N$ on the real λ axis.

Of physical interest is the effect of the singularities upon the spectrum for real values of λ. The coalescence of two adjacent levels at the complex values λ_c and $\bar{\lambda}_c$ signals avoided level crossing at $\lambda\approx\mathrm{Re}\lambda_c$. The further away the next pair of singularities is, the better this approximate relationship is obeyed.[12] It is exact for $N=2$ no matter how far away λ_c from the real axis. In other words isolated singularities behave locally like a two-dimensional problem. For $N>2$ it is important to keep in mind the sheet structure connecting the N levels: when two exceptional points $\lambda_c^{(1)}$ and $\lambda_c^{(2)}$ are near to each other or even coincide, their mutual influence with regard to the spectrum is immaterial if they lie in separate sheets. If E_i and E_{i+1} coalesce at $\lambda_c^{(1)}$ and E_k and E_{k+1} at $\lambda_c^{(2)}$, $i\neq k$, the two have no effect on each other even if $\lambda_c^{(1)}=\lambda_c^{(2)}$. Hence only if they lie in the same sheet and are near to each other, can they have an effect on avoided level crossing in that the relationship $\lambda\approx\mathrm{Re}\lambda_c$ is invalidated. In this case it is necessary and usually sufficient, to study the appropriate three-dimensional situation which is locally equivalent.[12] To ascertain the pair (E_i,E_k) that is connected at a particular branch point one has to follow a path in the λ plane from the real axis to that particular exceptional point where only two of the E_n will coalesce. While the pair (i,k) so determined will depend on the path chosen,[12] the path of physical interest is that of least distance to the real axis. In this way, the shape of the spectrum as a whole and, in particular, its statistical properties are determined by the positions of the exceptional points together with the corresponding sheet structure.

There must be strong correlations among the $N(N-1)/2$ complex conjugate pairs of exceptional points. While their positions are determined by $N(N-1)$ real parameters, the problem as such contains only $N^2/2+3N/2$ parameters [N from H_0 and $N+N(N-1)/2$ from H_1]. As a consequence, for sufficiently large N any perturbation of either H_0 or H_1 will affect the positions of all exceptional points, i.e., the spectrum as a whole: there is no combination of parameters that could affect one or only a few of the positions of the exceptional points. In the following section we shall see that many exceptional points can lie near to the real axis in models describing phase transitions. Motion of these singularities will cause great sensitivity of the anticrossing levels and corresponding state vectors since, by Eq. (5), the derivative of the anticrossing levels with respect to λ is infinity at λ_c and thus large in its vicinity. Anticipating at this stage our conjecture that the transitional region of a phase transition exhibits generically quantum chaos, we expect by this mechanism a high local sensitivity under perturbation as a signature of quantum chaos. In turn, the global statistical properties will be stable under perturbation.

Except for the last few remarks in the previous paragraph, the presentation of this section is general. In Sec. III we illustrate the relevance of our insights for the occurrence or absence of quantum chaos.

III. EXAMPLES

A. The Lipkin model perturbed

As one demonstration of physical interest we present results obtained from a particular perturbation of the Lipkin model.[7] In its simplest form the Lipkin Hamiltonian reads

$$H_\lambda = J_z - \frac{\lambda}{N-1}(J_x^2 - J_y^2) \tag{6}$$

with $J_i, i=x,y,z$ being the N-dimensional representations of the SU(2) operators. The Lipkin model is of the form $H_0+\lambda H_1$ with H_1 being a tridiagonal matrix. The model simulates nicely the essentials of a phase transition of a finite system. The spectrum shows no fluctuations[10] after proper unfolding, not even in the transitional region which lies at about $\lambda=1.0$ for sufficiently large N. The larger N, the narrower the transitional region. We now investigate instead the modified problem $H_\lambda = H_0+\lambda\hat{H}_1$ where $\hat{H}_1 = UH_1U^{-1}$ with U orthogonal. This modification leaves the spectrum unchanged before the phase transition ($\lambda=0$ and vicinity) and after it ($\lambda \gg 1.0$). The eigenfunctions before the phase transition are left virtually unchanged while after the phase transition they are subjected to the transformation by U.

The physical motivation for the particular modification of the model lies in the turning on of couplings between states that are uncoupled in the plain model, i.e., in admitting higher correlations. In a sense, we make the model more realistic by the transformation $H_1 \to \hat{H}_1$. Since, for the purpose of this paper, there is no point in trying to determine values of such coupling matrix elements, we chose them at random. In other words, the $N(N-1)/2$ angles which determine U are taken at random from the interval $[-\phi_0,\phi_0]$. Since the perturbation is assumed to be small, the value of ϕ_0 must be small. It turns out that $N\phi_0^2<0.1$ suffices to ensure that the overlap $\langle U\psi_i|\psi_i\rangle > 0.9$ when ψ_i is the ith eigenvector of H_1. This overlap indicates the extent to which the state vectors are modified by the perturbation beyond the phase transition ($\lambda \gg 1$).

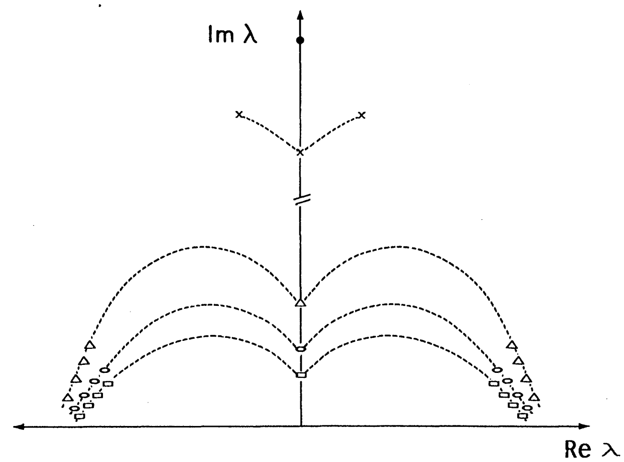

FIG. 1. Schematic representation of the positions of the exceptional points of the plain N-level Lipkin model (further explanation in text, Sec. III A).

For demonstration we illustrate schematically, in Fig. 1, the exceptional points of the unperturbed Lipkin model. The high symmetry of the plain Lipkin model is reflected in the symmetrical arrangement of the exceptional points: each exceptional point denoted in the plane represents the coalescence of two singularities which, however, occur in different Riemann sheets. On the imaginary λ axis each point represents only one exceptional point. We discern $N/2$ groupings of exceptional points. Starting from the real λ axis and moving to the imaginary λ axis, the first group (squares) connects the levels (1,2) and $(N-1,N)$, (2,3) and $(N-2,N-1),\ldots,(N/2,(N/2)+1)$. The second set (circles) connects the levels (2,3) and $(N-2,N-1),\ldots,(N/2,(N/2)+1)$ and so on until we reach the $N/2$th group which consists of one exceptional point connecting the levels $(N/2,(N/2)+1)$ (dot). In Fig. 2 trajectories are plotted for $N=6$ when a particular random set of the 15 angles is turned on from zero to $\phi_0=0.2$. While the $U=1$ the exceptional points are positioned very symmetrically, after perturbation the endpoints appear scattered around. This is reflected in the change of the spectra. We show in Figs. 3(a) and 3(b) the lower levels of the spectra for $N=400$, when $U=1$ and $\phi_0=0.015$, respectively. Clearly the spectrum is strongly affected in the transitional region. We recall that it is not affected far beyond and, as can be seen, at and around $\lambda=0$. The state vectors are likewise virtually unaffected in those regions while they are strongly affected in the transitional region.

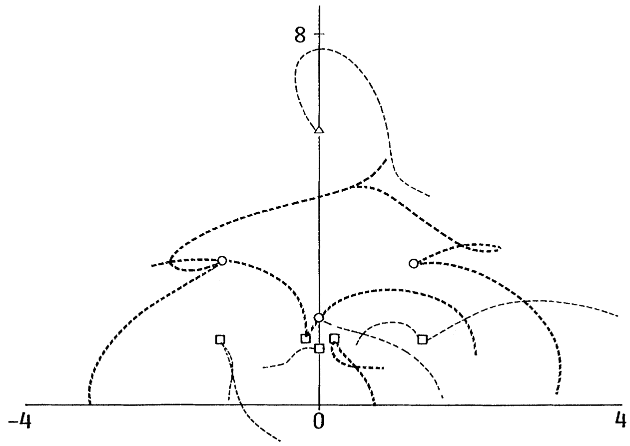

FIG. 2. Exceptional point trajectories for the Lipkin model ($N=6$) when a random set of angles is turned on from zero to $\phi_0=0.2$. The first group of unperturbed exceptional points ($U=1$) connect levels (1,2) and (5,6), (2,3) and (4,5), and (3,4) on the imaginary λ axis; the trajectories begin at the squares. The second group (circles) connects levels (2,3) and (4,5), and (3,4). The third group (triangle) connects levels (3,4).

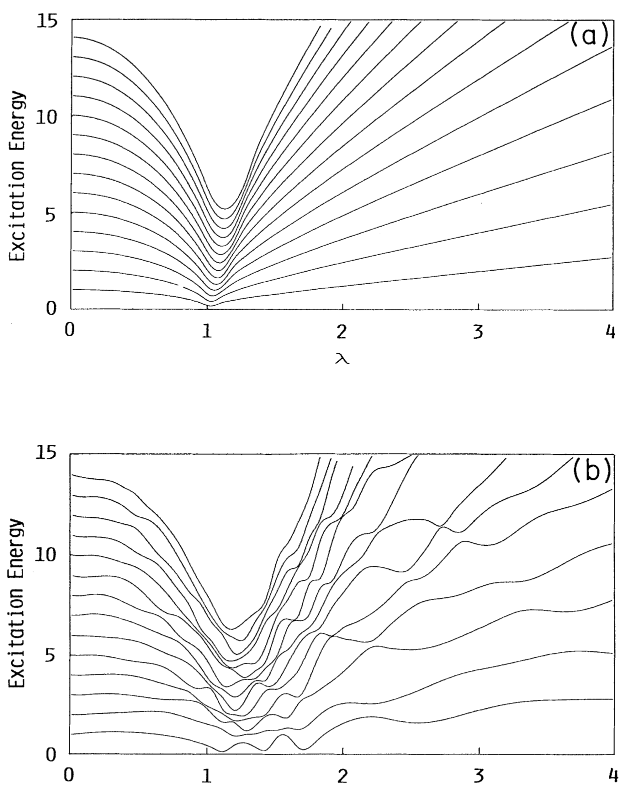

FIG. 3. (a) The lower end of the spectrum of the plain Lipkin model for $N=400$. The ground-state energy has been set equal to zero. (b) The lower end of the spectrum for the perturbed Lipkin model using random angles ϕ_0 from the interval $[-0.015,0.015]$. The spectrum remains unaffected in the vicinity of $\lambda=0.0$ and for $\lambda \gg 2.0$.

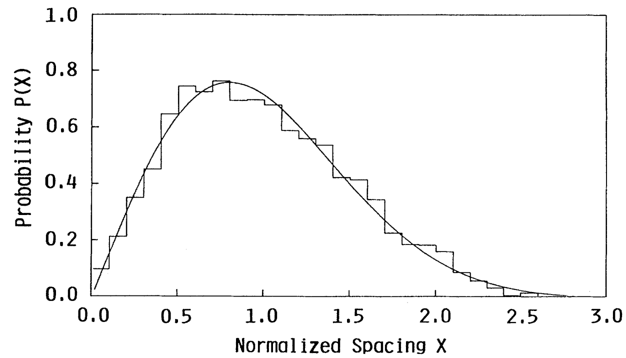

FIG. 4. The nearest-neighbor distribution for the perturbed Lipkin model. The data present an average over ten λ values in the transitional region. The data are in good agreement with the GOE statistic.

While the spectrum of the plain Lipkin model is strictly harmonic after proper unfolding, the perturbed model exhibits, in the transitional region, fluctuations characteristic of quantum chaos. In Figs. 4 and 5 the nearest-neighbor distribution and the $\Delta_3(L)$ function are displayed. The results are obtained as average values of ten λ values in the transition region between 1.04 and 1.76. The averages are taken to improve the statistics. In this context we report that the correlation between spectra for two different λ values decays rapidly with increasing distance of the λ values; the correlation length is about 0.02 for the λ range considered. In support of our claim that the level statistics are of the GOE type, i.e., the statistic is independent of λ over the λ range considered, we compared the second and third moment of the level distances and found them in good agreement within the statistical error. Here we encounter the remarkable pattern that on the one hand the energy levels E_n, and hence the spacings $D_n(\lambda)=E_{n+1}(\lambda)-E_n(\lambda)$, are analytic functions of λ which are connected to each other by the exceptional points. Thus, there is a mathematical relation between $D_n(\lambda)$ and $D_n(\lambda+\Delta\lambda)$. On the other hand the statistical evidence, as established by the traditional correlation function, yields a rapid decay of the correlation with increasing $\Delta\lambda$. This is brought about by

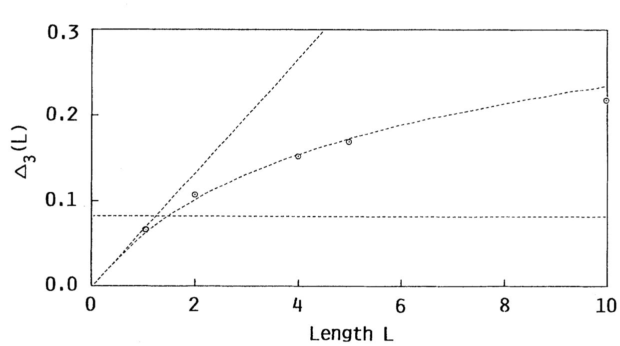

FIG. 5. The $\Delta_3(L)$ values for the perturbed Lipkin model ($N=400$). Note the good agreement with the $\Delta_3(L)$ function of the GOE.

the seemingly random distribution of the exceptional points producing the spectrum for real λ. We believe that this is the basic mathematical mechanism giving rise to quantum chaos.

B. Two simplified models.

In the following we report results of two matrix models for which, in contrast to the Lipkin model, the unperturbed problem is trivial in that H_1 is chosen diagonal. This means that there are no exceptional points or, we may say, all complex conjugate singularities coalesce on the real axis. The unperturbed problems are chosen in such a way that the spectrum is harmonic in the one case and obeys Poisson statistics in the other.[17]

1. Beginning with a harmonic spectrum

For N even, the unperturbed spectrum is generated by

$$H_0 = k\delta_{k,k'}$$

and (7)

$$H_1 = \left[\frac{N}{2} - k\right]\delta_{k,k'}, \quad k = 1, \ldots, N$$

which yields the spectrum $E_k^{(0)} = k(1-\lambda) + \lambda(N/2)$ for $\lambda < 1$ and $E_{N+1-k}^{(0)} = k(1-\lambda) + \lambda(N/2)$ for $\lambda > 1$ as illustrated in Fig. 6(a). The ith components of the eigenvectors are $|k\rangle_i = \delta_{i,k}$ for $\lambda < 1$ and $|k\rangle_i = \delta_{i,N+1-k}$ for $\lambda > 1$. Similar to the procedure above we now consider instead $H_\lambda = H_0 + \lambda \hat{H}_1$. The spectrum of this perturbed problem looks qualitatively like the one of Fig. 3(b) and the statistical properties are the same. The difference between this and the perturbed Lipkin model lies in the much higher sensitivity of the present case. The value $\phi_0 = 0.003$ is sufficient to turn the single line in the next-nearest neighbor distribution into a Wigner curve. Accordingly tiny are the changes upon the state vectors for $\lambda \gg 1$. For $N = 400$ and ϕ_0 as indicated $U\psi_i$ and ψ_i are indistinguishable for practical purposes. It should be noted that in the limit of very large N, the spectrum of the Lipkin model of Sec. III A becomes more and more similar to the present matrix model in its unperturbed form. This is related to the fact that the exceptional points of the unperturbed Lipkin model move towards the point $\lambda = 1$ for $N \to \infty$. In the present model the exceptional points emerge from this point under peturbation.

2. Beginning with a spectrum obeying Poisson statistics

In this example we generate an unperturbed spectrum, illustrated in Fig. 6(b), which obeys the Poisson statistic for the nearest-neighbor distribution. We choose

$$H_0 = k\delta_{k,k'}$$

and (8)

$$H_1 = \mathcal{P}\left[\frac{N}{2} - k\right]\delta_{k,k'}, \quad k = 1, \ldots, N$$

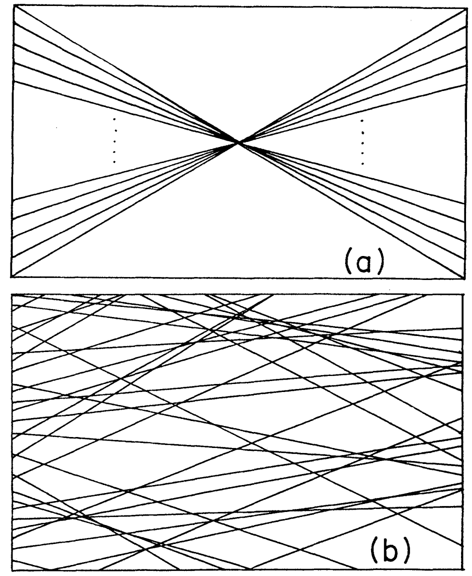

FIG. 6. (a) The unperturbed spectrum of the model in Sec. III B 1. In contrast to 3(a) and 3(b) the actual spectrum is displayed in this and the figure below. (b) A section of the unperturbed spectrum of the model in Sec. III B 2.

where $\mathcal{P}(k)$ is a random permutation of the integers k. The spectrum is $E_k^{(0)} = o(k + \lambda\mathcal{P}[(N/2) - k])$ where we denote by $o(n(k))$ the ordered sequence of the numbers $n(k)$. The nearest-neighbor distribution of the energies is a Poisson distribution. The eigenvectors are all of the form $(0 \cdots 1 \cdots 0)$ but their sequence is randomly distributed. Only for $\lambda > \lambda_{\max}$ are they ordered as in Sec. III A 1 and given by $|k\rangle = \delta_{i,N+1-k}$; by $\lambda_{\max}$ we denote the most right intersection point of the levels. Again we consider the modified Hamiltonian $H_\lambda = H_0 + \lambda \hat{H}_1$. It is no surprise that the range of λ values where avoided level crossing occurs is now more extended than in the previous examples and essentially ranges from $\lambda = 0$ to $\lambda = \lambda_{\max}$. The nearest-neighbor distribution for $\lambda = 1$ and $\phi_0 = 0.05$ is in good agreement with the GOE statistic for $N = 400$. For smaller values of ϕ_0 the distribution lies between the Poisson and Wigner curve. Note that the value chosen for ϕ_0 implies that the state vectors are changed significantly also for $\lambda \gg \lambda_{\max}$, in contrast to the previous examples. We will return to this point in Sec. V.

IV. MEAN FIELDS AND OTHER APPROXIMATION METHODS

The Lipkin model and its extensions have served as a testing ground for various approximation methods.[18,19] It is accepted that the mean-field approach with or without projection performs well outside, but not within the transitional region.[20] This applies equally to the approximate calculation of excited states via the random-phase approximation.

The Hartree-Fock ground state in the Lipkin model is given by

$$|\psi_{\rm HF}\rangle = e^{2i\gamma J_y}|0\rangle \,, \tag{9}$$

where $|0\rangle$ is the noninteracting ground state, $\cos 2\lambda = 1$ for $\lambda < 1$ and $\cos 2\lambda = 1/\lambda$ for $\lambda \ge 1$. These are the values that minimize

$$E_{\rm HF}(\gamma) = \langle \psi_{\rm HF}|H_\lambda|\psi_{\rm HF}\rangle = -\frac{N}{2}\cos 2\gamma - \frac{\lambda N}{4}(1-\cos^2 2\gamma) \tag{10}$$

with H_λ given by Eq. (6). It is expected that the Hartree-Fock approximation is not greatly affected by the perturbation $H_1 \to UH_1U^{-1}$ outside the transitional region as long as the angles in U are sufficiently small. Explicit expressions can be obtained for the energy to first order in the angles. Up to linear terms in the angles we have the perturbed Hamiltonian

$$H_\lambda = J_z - \frac{\lambda}{N-1}\{(J_x^2 - J_y^2) + i\boldsymbol{\phi}\cdot[\mathbf{R}, J_x^2 - J_y^2]\} \tag{11}$$

where the vector product runs over all the $N(N-1)/2$ angles associated with the corresponding generators in N dimensions

$$iR_{k,l}^{s,t} = \delta_{k,s}\delta_{l,t} - \delta_{k,t}\delta_{l,s}$$
$$s = 1, \ldots, N-1, \quad t = s+1, \ldots, N \tag{12}$$

and [,] denotes the commutator. To get the additional Hartree-Fock contribution we have to sandwich the commutator between the Hartree-Fock state. After some algebra we obtain

$$i\langle \psi_{\rm HF}|[R^{s,t}, J_x^2 - J_y^2]|\psi_{\rm HF}\rangle = 2\sqrt{N}(\cos^2\gamma - \sin^2\gamma)\cos\gamma\sin\gamma[(e^{-2i\gamma J_y})_{1,s}(e^{2i\gamma J_y})_{t,2} - (s \leftrightarrow t)]$$
$$+ 2\sqrt{2N(N+1)}\cos^2\gamma[(e^{-2i\gamma J_y})_{1,s}(e^{2i\gamma J_y})_{t,3} - (s \leftrightarrow t)] \,. \tag{13}$$

The N-dimensional representation of the matrices $\exp(2i\gamma J_y)$ are well known,[21] we have for instance

$$(e^{-2i\gamma J_y})_{1,s} = (e^{2i\gamma J_y})_{s,1} = \left[\frac{(N-1)!}{(s-1)!(N-s)!}\right]^{1/2}(\cos\gamma)^{s-1}(\sin\gamma)^{N-s} \,. \tag{14}$$

The $(2,s)$ and $(3,s)$ elements are likewise polynomials in the trigonometric functions with two and three terms, respectively. The additional contribution towards the Hartree-Fock energy is minute for two reasons: (i) The powers of $\cos\gamma$ are nonzero ($s > 1$) which can give a substantial contribution from $\exp(2i\gamma J_y)$ only for $\gamma \approx 0$; (ii) the angles $\phi^{s,t}$ are chosen to be small. As a consequence, for $\lambda > 1$ where $\gamma > 0$ the HF contribution is immaterial; for $\lambda < 1$ the solution for the energy minimization yields again $\cos\gamma > 1$ as it does for the unperturbed case where all $\phi^{s,t}$ vanish. We conclude that only at the transitional point could the additional term have a possible influence upon the solution; it is not expected that this can improve the quality of the mean-field approximation. The crucial point is that before and beyond the phase transition the mean field, and thus also its performance, remains unaffected. This applies also to excited states when calculated via the random-phase approximation. It confirms the fact that the order prevailing in these regions is not disturbed by the perturbation while, in contrast, the high instability against perturbation is manifested in the transitional region where the mean-field approximation fails.

We mention the more involved generator coordinate method. It can be implemented exactly[18] for the plain Lipkin model because of the high inherent symmetry of the model. The symmetry is also reflected in the pattern of the exceptional points as sketched in Fig. 1. Note that the perturbation completely destroys the symmetric pattern of the exceptional points and it is obvious that all the advantages of the method disappear by the perturbation.

We belive that our findings are symptomatic for nuclear structure calculations. The success of approximation methods including the concept of collective states is due to an order that is stable under perturbation. This situation is associated with exceptional points which are remote from the parameter range of interest. In turn, a transitional region is necessarily connected with many exceptional points in the immediate vicinity of the real parameter values under consideration. They bring about high sensitivity under perturbation. The instability can, in principle, be so pronounced that only statistical information can be extracted eventually from spectra and matrix elements as it is this type of information that is stable: this is the signature of quantum chaos. In this context we stress that the regular behavior of the plain Lipkin model is similarly nongeneric as is, say, the classical harmonic oscillator in more than one dimension. A generic perturbation of either case leads into the chaotic regime. Yet there is an important difference: the quantum-mechanical transition from regular to chaotic is smooth, albeit dramatic, when the perturbation is turned on; classical systems switch from chaotic to regular behavior usually at isolated points of the relevant parameter.

V. EIGENSTATES

The effect of the exceptional points upon the spectrum is reflected in the effect upon the eigenstates. Let us denote by $\mathcal{W}_{m,n}(\lambda)$ the orthogonal matrix which diagonalizes H_λ. It is an N-dimensional rotation which is characterized by $N(N-1)/2$ angles. Clearly, $\mathcal{W}(0)$ is the unit matrix while $\mathcal{W}(\infty)$ diagonalizes H_1. The column vectors of $W(\lambda)$ are the eigenvectors of the problem.

A level repulsion in a two-dimensional matrix model is associated with an interchange (and sign change) of the two corresponding eigenvectors. In an N-dimensional situation the effect is essentially the same as long as the complex conjugate pair of exceptional points is sufficiently distant from further singularities. It can be viewed as a rotation of the matrix $\mathcal{W}$ affecting only two axes in an N-dimensional space where the angle of the rotation is $\leq|\pi/2|$. When λ changes from 0 to ∞ the axis structure (by which we may visualize $\mathcal{W}$) moves from the initial orientation $\mathcal{W}(0)$ to its final orientation $\mathcal{W}(\infty)$ in a particular way. In general, all possible combinations of axes are involved in that motion.

Correlation coefficients of the form

$$\frac{\sum_{m,n}^{N}\mathcal{W}_{m,n}(\lambda)W_{m,n}(\lambda+\Delta\lambda)-\frac{1}{N}\left[\sum_{m,n}^{n}\mathcal{W}_{m,n}(\lambda)\right]^2}{\sum_{m,n}^{N}\mathcal{W}_{m,n}(\lambda)\mathcal{W}_{m,n}(\lambda)-\frac{1}{N}\left[\sum_{m,n}^{N}\mathcal{W}_{m,n}(\lambda)\right]^2}$$

were calculated and a pattern similar to that of the corresponding spectrum was found: i.e., if there is a fast decay of correlations for successive λ values in the spectrum, the same is found for the associated orientations of $\mathcal{W}(\lambda)$ for which the correlation length is 0.005. This is a reflection of a "whirling" motion of the orthogonal axes under variation of λ. In the transitional region, the correlation coefficient of $\mathcal{W}_{m,n}(\lambda)$ is on average ten times larger for the plain Lipkin model than for the perturbed Lipkin model ($N=400$). In the latter case this corresponds to a gentle motion without "wiggles" which is describable by much fewer parameters as is reflected by the high [SU(2)] symmetry of the plain model. Outside the transitional region the spectrum and the corresponding eigenvectors remain appreciably correlated for both the plain and the perturbed Lipkin models.

In Secs. III A and III B 1 the perturbation needed to produce occurrence of quantum chaos in the transition region was so small that the eigenvectors remained essentially unchanged for $\lambda \gg 1$. (Note, that they are strongly affected as λ is sweeping over the transition region.) In other words $\mathcal{W}(\infty)$ is almost identical for the plain and perturbed cases. In the mathematical model in Sec. III B 2, where the unperturbed model obeys Poisson statistics, the transitional region is drawn out and is much more robust to perturbation. For sufficiently large values of the angles the level statistics in the transitional region also obey the Wigner surmise, yet in this case, in order to obtain this result, $\mathcal{W}(\infty)$ is significantly different for the plain and perturbed models. The difference between these two cases in their response to perturbation lies in the density of the exceptional points. In the first instance (Secs. III A and III B 1), the density of the exceptional points within the transitional region is large and hence there is an extreme sensitivity under perturbation due to mutual influence of the singularities upon one another. In the second case the exceptional points are less densely distributed since the parameter range of interest is now extended and thus the effects of surrounding singularities become less important. While the local sensitivity to perturbation remains unchanged, the global sensitivity is now reduced with respect to the first instance.

In all cases the level statistics remain unaffected irrespective of a further increase in the angles in U, once the GOE statistics have been attained. Of course the detailed pattern of the spectrum does depend on the precise form of U, in fact it is a smooth function of the angles.

VI. SUMMARY AND DISCUSSION

Quantum chaos is associated with avoided level crossings. Level repulsion in turn is associated with the singularities of the spectrum $E_n(\lambda)$, namely, the exceptional points of the Hamiltonian H_λ. In this paper we investigate Hamiltonians of the form $H(\lambda)=H_0+\lambda H_1$ where either the symmetries of H_0 and H_1 are incompatible or the two operators do not commute. For a certain λ range a transitional region is found. Here avoided level crossing occurs on a large scale and therefore there are also many exceptional points. Generically phase transition regions are associated with increased level density, this affects, in particular, the region of our interest, i.e., the lower end of the spectrum. We conjecture that it is the distribution of the exceptional points that gives rise to the occurrence or absence of quantum chaos in the transitional region. Although no quantitative statements are made in this paper about the actual distribution of the singularities, there appear to be indications that the detailed nature of the distribution is of lesser importance. Further investigations focusing, in particular, on the change from order to chaos, when the angles of U are turned on, are expected to shed more light onto this aspect. It should be noted that determining the precise locations of the exceptional points is practically impossible for large systems. However, first attempts have been made to determine their distribution.[9]

Insight has been gained about the local sensitivity of the spectrum and corresponding eigenvectors in terms of the motion of closely lying singularities. Strong correlations exist between the $N(N-1)/2$ complex conjugate pairs of exceptional points. As a consequence, for sufficiently large N, the spectrum is affected by any perturbation of H_0 or H_1. The transitional region is characterized by many exceptional points close to the real λ axis. Any motion of the exceptional points cause great sensitivity of the anticrossing levels and corresponding wave functions since, as seen in Sec. III, the derivative of the anticrossing levels with respect to λ is infinity at λ_c and thus the variation of the spectrum and wave function is strongly affected in its vicinity. We conjecture that the high local sensitivity under perturbation in the phase transition region is a signature of quantum chaos. A generic perturbation of the system will cause a redistribution of the exceptional points, but will not necessarily change the global statistical properties except for a situation like the plain Lipkin model. In this context, the absence of GOE statistics in the transition region of the plain Lipkin model must be seen as an isolated case reminiscent of the nongenericity of regular behavior of, say, the classical harmonic oscillator in more than one dimension. Note, however, that sensitivity prevails in the plain as well as in the perturbed Lipkin model.

The examples in Sec. III highlight the relationship between sensitivity to perturbation and the energy-level density in the spectrum. It is known[22] that the sensitivity depends on the ratio of the average coupling matrix element and the mean level spacing. In Sec. II B 1 at one value of λ the mean level spacing is zero in the unperturbed problem and hence the sensitivity to perturbation is infinite. In Sec. III B 2 the average level spacing is finite and constant; accordingly, a comparatively larger perturbation is required to obtain GOE statistics in the transitional region. This can be understood in terms of the accumulation of exceptional points in a narrow window of the λ range in Sec. III B 1. Due to mutual influence of closely lying singularities, there is an enhanced global sensitivity to perturbation. In contrast, the exceptional point distribution is diluted in Sec. III B 2, hence there is little mutual influence. Although local sensitivity to perturbation is retained the overall, global sensitivity is reduced and a larger set of angles is required to produce statistics obeying the Wigner surmise.

In conclusion, the mathematical mechanism connecting the level repulsion and the dramatic change of the ground-state and excited state structure present in the phase transition region lies in the exceptional points of the Hamiltonian. In this region where the mean-field approximation fails, the positions of many exceptional points in the parameter range of interest are highly sensitive to perturbation. We conjecture that the signature of quantum chaos is the resultant pronounced instability under perturbation, which results in statistical information being the only stable information drawn from the spectra and matrix elements.

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