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Rational approximants using Levin–Weniger transforms

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Abstract

The existing work on generating rational approximants of functions from their series expansions is extended to include the generalization of the Levin transforms due to Weniger. It is seen that this leads to approximants even better than the u-approximants obtained previously. It is further seen that the freedom of choosing an additional parameter β in these transforms can be exploited to dramatically increase their effectiveness in certain situations.

Keywords: Series tranformation; Rational approximants; Divergent-series; Levin–Weniger transforms; Padé approximation

1. Introduction

Infinite series, either slowly convergent or even divergent, occur frequently in the mathematical analysis of scientific problems like the solution of a differential equation, an integral equation or in perturbation theory. In many scientific problems the computation of larger and larger numbers of terms of the series becomes increasingly difficult and, in many cases, the higher order terms are affected by serious inaccuracies. Because of this, one is confronted with the problem of estimating a function from a few terms of its series representation.

It is well known that nonlinear sequence transforms are very effective accelerators of convergence on monotone and alternating sequences of numbers. Moreover, these sequence transforms induce convergence in divergent sequences and are valid methods of summation. Probably the oldest nonlinear transform is the famous Aitken Δ^2-transform [1] and the current interest in nonlinear sequence transforms stems originally from the articles on the e-transform and the ε-transform by Shanks [2] and Wynn [3] respectively. A survey of the nonlinear transforms can be obtained in the review article by Weniger [4] and in the book by Brezinski and Zaglia [5]. A unified discussion of the different nonlinear transforms can also be found in the article by Bhowmick et al. [6].

When a nonlinear sequence transform is applied to the sequence of partial sums of a power series, it generates approximants in the form of rational functions. The Padé approximants, which are rational approximants derived from the power series, have been frequently used in tackling divergent series encountered in theoretical physics [7] and in simulating analytic properties of the function such as zeros or singularities that are not apparent from the series expansion. Active research on nonlinear sequence transforms contributed significantly not only to the understanding of the relation between the Padé approximants and the ε-transform, but also led to the discovery of other sequence transforms. It has been demonstrated by Smith and Ford [8,9] and

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Bhowmick et al. [6] that the θ-transform of Brezinski [10] and the Levin tranforms [11] are move effective than the ε-transform in summing a wide class of convergent and divergent test sequences of numbers, both real and complex. It is thus quite natural to ask how the approximants of a power series built with these sequence transforms compare with the Padé approximants. Recently it has been shown by Roy et al. [12] that the fidelity of the rational approximants built with the Levin u-transform is in general better than the standard Padé approximants except for functions which have both poles and zeros. Weniger [4] has made a number of generalizations of the Levin transform assuming that the remainder estimate can be expressed as a factorial series instead of a power series as in the case of a Levin transform. The purpose of the present paper is to make a comparative study of the different approximants built with such generalized transforms (henceforth called the Weniger transforms) with Padé approximants.

We also examine the relative performance of these transforms over a large range of the argument of the approximants obtained from them as opposed to their effectiveness on sequences of numbers which simply correspond to specific points in this range. As we shall see, the relative effectiveness of the transforms varies over the range considered, and conclusions from studies only on sequences of numbers can, therefore, be generalized.

In the next section we briefly introduce the Padé approximant. In the subsequent section we introduce the Weniger transforms and the associated approximants. In Section 4 we compare the performance of these Weniger approximants with the Padé approximants on a number of test series which include convergent, divergent and asymptotic series. In the final section we discuss some interesting findings on the Weniger transforms.

2. Padé approximants and the ε-transform

Let us assume that $f(z)$ is analytic in the neighbourhood of $z=0$, so that we can write

$$f(z)=\sum_{n=0}^{\infty} a_n z^n. \tag{1}$$

The Padé approximants $[m,n]$ of $f(z)$ are uniquely determined by the rational functions defined by

$$[m,n]=\frac{P_m(x)}{Q_n(x)}, \tag{2}$$

where $P_m(z)$ and $Q_n(z)$ are polynomials in z of degree m and n respectively, such that for any pair of integers (m,n)

$$f(z)-[m,n]=\mathrm{O}(z^{m+n+1}).$$

The Padé approximants $[m,n]$ form a two-dimensional array called the Padé table. The ε-transform, which is closely related to the Padé table, may be stated in the following form.

If $\{S_n,\ n=1,2\ldots\}$ be the sequence of partial sums of the series given by Eq. (1), i.e.

$$S_n=\sum_{k=0}^{n-1} a_k z^k,\quad n=1,2\ldots, \tag{3}$$

then we define

$$\varepsilon_0^{(n)}=S_n,\qquad \varepsilon_{-1}^{(n)}=0\qquad n=1,2,\ldots,$$
$$\varepsilon_{k+1}^{(n)}=\varepsilon_{k-1}^{(n+1)}+\frac{1}{\varepsilon_k^{(n+1)}-\varepsilon_k^{(n)}}\quad k=0,1,\ldots. \tag{4}$$

The ε's with odd lower index are intermediate quantities without any interesting meaning and only the even order ε's are meaningful. It can be shown that

$$\varepsilon_{2k}^{(n)} = [n+k,k]. \tag{5}$$

Thus the recursive ε-algorithm produces the Padé approximants and the convergence theory of the Padé approximants [7,13] can be applied to the ε-algorithm.

3. Approximants with Weniger transforms

If $\{S_n,\ n=1,2,\dots\}$ is the sequence of partial sums of a power series defined by Eq. (3), then one can define an associated sequence $\{g_n\}$ by the relation

$$S = S_n + g_n \omega_n, \tag{6}$$

where the ω_n's are known functions of n which are different from zero and distinct for all finite values of n and

$$S = f(z) = \sum_{i=0}^{\infty} a_i z^i. \tag{7}$$

The Levin sequence transform is designed to be exact for model sequences of the type

$$S - S_n = \omega_n \sum_{i=0}^{k-1} \frac{c_i}{(n+\beta)^i}, \quad k = 1,2,\dots \tag{8}$$

and serve as a finite approximation to a Poincaré-type asymptotic expansion of the following type:

$$S - S_n = \omega_n \sum_{i=0}^{\infty} \frac{c_i}{(n+\beta)^i}. \tag{9}$$

Eq. (8) can be rewritten in the following form:

$$\frac{(S-S_n)(n+\beta)^{k-1}}{\omega_n} = \sum_{i=0}^{k-1} c_i (n+\beta)^{k-i-1} = p_{k-1}^{(n)}, \tag{10}$$

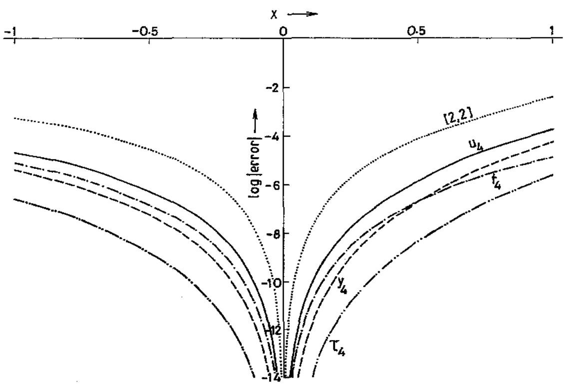

Fig. 1. Plot of the logarithm of the absolute error for e^x against x using different approximants of order 4 with $\beta = 0$.

where $p_{k-1}^{(n)}$ is a polynomial in n of order $k-1$. Using the fact that any polynomial of degree $k-1$ can be annihilated by the forward difference operator Δ^k defined as

$$\Delta^k f(n) = \Delta^{k-1} f(n+1) - \Delta^{k-1} f(n), \quad \Delta^0 f(n) = f(n), \tag{11}$$

we can write

$$\Delta^k p_{k-1}(n) = 0 = \Delta^k \left(\frac{(S-S_n)(n+\beta)^{k-1}}{\omega_n} \right). \tag{12}$$

Table 1
Different approximations of order 4 for different test functions $f(x)$

1. $f(x) = e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!}$

$$u_4 = \frac{24(5+\beta)^2 + 24(9+2\beta)x + 24x^2}{24(5+\beta)^2 - 24(4+\beta)^2 x + 12(3+\beta)^2 x^2 - 4(2+\beta)^2 x^3 + (1+\beta)^2 x^4}$$

$$t_4 = \frac{24(5+\beta)^3 + 24(61+27\beta+3\beta^2)x + 72(4+\beta)x^2 + 24x^3}{24(5+\beta)^3 - 24(4+\beta)^3 x + 12(3+\beta)^3 x^2 - 4(2+\beta)^3 x^3 + (1+\beta)^3 x^4}$$

$$y_4 = \frac{24(5+\beta)_{(2)} + 48(5+\beta)x + 24x^2}{24(5+\beta)_{(2)} - 24(4+\beta)_{(2)} x + 12(3+\beta)_{(2)} x^2 - 4(2+\beta)_{(2)} x^3 + (1+\beta)_{(2)} x^4}$$

$$\tau_4 = \frac{24(5+\beta)_{(3)} + 72(5+\beta)_{(2)} x + 72(5+\beta)x^2 + 24x^3}{24(5+\beta)_{(3)} - 24(4+\beta)_{(3)} x + 12(3+\beta)_{(3)} x^2 - 4(2+\beta)_{(3)} x^3 + (1+\beta)_{(3)} x^4}$$

2. $f(x) = \log(1+x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{m+1}}{m+1}$

$$u_4 = \frac{60(5+\beta)^2 x + 6(387+206\beta+27\beta^2)x^2 + 4(227+182\beta+35\beta^2)x^3 + (1+\beta)(61+37\beta)x^4}{60(5+\beta)^2 + 192(4+\beta)^2 x + 216(3+\beta)^2 x^3 + 96(2+\beta)^2 x^3 + 12(1+\beta)^2 x^4}$$

$$t_4 = \frac{60(5+\beta)^3 x + 6(1423+1161\beta+309\beta^2+27\beta^3)x^2 + 4(547+681\beta+273\beta^2+35\beta^3)x^3 + (1+\beta)^2(73+37\beta)x^4}{60(5+\beta)^3 + 192(4+\beta)^3 x + 216(3+\beta)^3 x^2 + 96(2+\beta)^3 x^3 + 12(1+\beta)^3 x^4}$$

$$y_4 = \frac{60(5+\beta)_{(2)} x + 6(5+\beta)(98+27\beta)x^2 + 4(318+217\beta+35\beta^2)x^3 + (110+135\beta+37\beta^2)x^4}{60(5+\beta)_{(2)} + 192(4+\beta)_{(2)} x + 216(3+\beta)_{(2)} x^2 + 96(2+\beta)_{(2)} x^3 + 12(11+\beta)_{(2)} x^4}$$

$$\tau_4 = \frac{60(5+\beta)_{(3)} x + 18(5+\beta)_{(2)}(31+9\beta)x^2 + 4(5+\beta)(282+203\beta+35\beta^2)x^3 + (354+551\beta+258\beta^2+37\beta^3)x^4}{60(5+\beta)_{(3)} + 192(4+\beta)_{(3)} x + 216(3+\beta)_{(3)} x^2 + 96(2+\beta)_{(3)} x^3 + 12(1+\beta)_{(3)} x^4}$$

3. $f(x) = \frac{\sin x}{x} = \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m+1)!}$

$$u_4 = \frac{362880(5+\beta)^2 - 20160(59+22\beta+2\beta^2)x^2 + 96(295+80\beta+4\beta^2)x^4 - 96x^6}{362880(5+\beta)^2 + 20160(4+\beta)^2 x^2 + 720(3+\beta)^2 x^4 + 24(2+\beta)^2 x^6 (1+\beta)^2 x^8}$$

$$t_4 = \frac{362880(5+\beta)^3 - 20160(311+177\beta+33\beta^2+2\beta^3)x^2 + 96(1900+885\beta+120\beta^2+4\beta^3)x^4 - 144(9+2\beta)x^6}{362880(5+\beta)^3 + 20160(4+\beta)^3 x^2 + 720(3+\beta)^3 x^4 + 24(2+\beta)^3 x^6 + (1+\beta)^3 x^8}$$

$$y_4 = \frac{362880(5+\beta)_{(2)} - 40320(5+\beta)(7+\beta)x^2 + 96(335+84\beta+4\beta^2)x^4 - 96x^6}{362880(5+\beta)_{(2)} + 20160(4+\beta)_{(2)} x^2 + 720(3+\beta)_{(2)} x^4 + 24(2+\beta)_{(2)} x^6 + (1+\beta)_{(2)} x^8}$$

$$\tau_4 = \frac{362880(5+\beta)_{(3)} - 20160(5+\beta)_{(2)}(17+2\beta)x^2 + 96(5+\beta)(573+112\beta+4\beta^2)x^4 - 144(11+2\beta)x^6}{362880(5+\beta)_{(3)} + 20160(4+\beta)_{(3)} x^2 + 720(3+\beta)_{(3)} x^4 + 24(2+\beta)_{(3)} x^6 + (1+\beta)_{(3)} x^8}$$

Table 1 *Continued*

4. $f(x) := \phi(x,1.1,0) = \sum_{m=1}^{\infty} \frac{x^m}{m^{1.1}}$

$u_4 = [5.87309(5+\beta)^2 x - 15.6393(3.37106+\beta)(4.27855+\beta)x^2 +$
$13.27(2.10523+\beta)(3.13122+\beta)x^3 - 3.41256(1.01474+\beta)(1.67888+\beta)x^4]/$
$[5.87309(5+\beta)^2 - 18.3792(4+\beta)^2 x + 20.0902(3+\beta)^2 x^2 - 8.57419(2+\beta)^2 x^3 + (1+\beta)^2 x^4]$

$t_4 = [5.87309(5+\beta)^3 x - 15.6393(2.87127+\beta)(18.5678+8.60315\beta+\beta^2)x^2 +$
$13.27(1.86304+\beta)(2.39435+\beta)(3.59729+\beta)x^3 -$
$3.41256(0.966389+\beta)(1.06556+\beta)(2.00848+\beta)x^4]/$
$[5.87309(5+\beta)^3 - 18.3792(4+\beta)^3 x + 20.0902(3+\beta)^3 x^2 - 8.57419(2+\beta)^3 x^3 + (1+\beta)^3 x^4]$

$y_4 = [5.87309(5+\beta)_{(2)} x - 15.6393(3.64961+\beta)(5+\beta)x^2 +$
$13.27(2.40187+\beta)(3.83458+\beta)x^3 - 3.41256(1.24658+\beta)(2.44703+\beta)x^4]/$
$[5.87309(5+\beta)_{(2)} - 18.3792(4+\beta)_{(2)} x + 20.0902(3+\beta)_{(2)} x^2 - 8.57419(2+\beta)_{(2)} x^3 +$
$(1+\beta)x^4]$

$\tau_4 = [5.87309(5+\beta)_{(3)} x - 15.6393(3.47442+\beta)(5+\beta)_{(2)} x^2 +$
$13.27(2.3294+\beta)(3.52528+\beta)(5+\beta)x^3 -$
$3.41256(1.22078+\beta)(2.2935+\beta)(3.52615+\beta)x^4]/$
$[5.87309(5+\beta)_{(3)} - 18.3792(4+\beta)_{(3)} x + 20.0902(3+\beta)_{(3)} x^2 -$
$8.57419(2+\beta)_{(3)} x^3 + (1+\beta)_{(3)} x^4]$

5. $f(x) = \int_0^{\infty} \frac{e^{-t}}{1+xt} dt = \sum_{m=0}^{\infty} m!(-x)^m$

$$u_4 = \frac{(5+\beta)^2 + (11+3\beta)(21+5\beta)x + 2(221+162\beta+29\beta^2)x^2 + 2(1+\beta)(49+25\beta)x^3}{(5+\beta)^2 + 16(4+\beta)^2 x + 72(3+\beta)^2 x + 96(2+\beta)^2 x^3 + 24(1+\beta)^2 x^4}$$

$$t_4 = \frac{(5+\beta)^3 + (899+693\beta+177\beta^2+15\beta^3)x + 2(3=\beta)(195+156\beta+29\beta^2)x^2 + 2(1+\beta)^2(61+25\beta)x^3}{(5+\beta)^3 + 16(4+\beta)^3 x + 72(3+\beta)^3 x^2 + 96(2+\beta)^3 x^3 + 24(1+\beta)^3 x^4}$$

$$y_4 = \frac{(5+\beta)(6+\beta) + (5+\beta)(58+15\beta)x + 2(302+191\beta+29\beta^2)x^2 + 2(86+99\beta+25\beta^2)x^3}{(5+\beta)_{(2)} + 16(4+\beta)_{(2)} x + 72(3+\beta)_{(2)} x^2 + 96(2+\beta)_{(2)} x^3 + 24(1+\beta)_{(2)} x^4}$$

$$t_4 = \frac{(5+\beta)_{(3)} + 3(5+\beta)_{(2)}(19+5\beta)x + 2(5+\beta)(282+185\beta+29\beta^2)x^2 + 2(282+419\beta+186\beta+25\beta^3)x^3}{(5+\beta)_{(3)} + 16(4+\beta)_{(3)} x + 72(3+\beta)_{(3)} x^2 + 96(2+\beta)_{(3)} x^3 + 24(1+\beta)_{(3)} x^3}$$

6. $f(x) = \int_x^{\infty} \frac{e^{-t} dt}{t} = E_1(x) = \frac{1}{xe^x} \sum_{m=0}^{\infty} (-1)^m m!\, x^{-m}$

$$u_4 = \frac{2(1+\beta)(49+25\beta)x + 2(221+162\beta+29\beta^2)x^2 + (11+3\beta)(21+5\beta)x^3 + (5+\beta)^2 x^4}{24(1+\beta)^2 + 96(2+\beta)^2 x + 72(3+\beta)^2 x^2 + 16(4+\beta)^2 x^3 + (5+\beta)^2 x^4}$$

$$t_4 = \frac{2(1+\beta)^2(61+25\beta)x + 2(3+\beta)(195+156\beta+29\beta^2)x^2 + (899+693\beta+177\beta^2+15\beta^3)x^3 + (5+\beta)^3 x^4}{24(1+\beta)^3 + 96(2+\beta)^3 x + 72(3+\beta)^3 x^2 + 16(4+\beta)^3 x^3 + (5+\beta)^3 x^4}$$

$$y_4 = \frac{2(86+99\beta+25\beta^2)x + 2(302+191\beta+29\beta^2)x^2 + (5+\beta)(58+15\beta)x^3 + (5+\beta)(6+\beta)x^4}{24(1+\beta)_{(2)} + 96(2+\beta)_{(2)} x + 72(3+\beta)_{(2)} x^2 + 16(4+\beta)_{(2)} x^3 + (5+\beta)_{(2)} x^4}$$

$$\tau_4 = \frac{2(282+419\beta+186\beta^2+25\beta^3)x + 2(5+\beta)(282+185\beta+29\beta^2)x^2 + 3(5+\beta)_{(2)}(19+5\beta)x^3 + (5+\beta)_{(3)} x^4}{24(1+\beta)_{(3)} + 96(2+\beta)_{(3)} x + 72(3+\beta)_{(3)} x^2 + 16(4+\beta)_{(3)} x^3 + (5+\beta)_{(3)} x^4}$$

This gives the kth order Levin-transform as an approximation to S, i.e.

$$S \approx \mathscr{L}_k^{(n)} = \frac{\Delta^k \left(\frac{(n+\beta)^{k-1} S_n}{\omega_n} \right)}{\Delta^k \left(\frac{(n+\beta)^{k-1}}{\omega_n} \right)}. \tag{13}$$

Levin [11] suggested

$$\omega_n = (\beta + n)\,\Delta S_{n-1}. \tag{14}$$

This amounts to assuming

$$\frac{S - S_n}{\Delta S_{n-1}} = \alpha(n+\beta) + \sum_{i=0}^{k-2} \frac{c_i}{(n+\beta)^i}, \tag{15}$$

which is the behaviour of a generalized zeta-like series.

With this choice of ω_n one gets the Levin u-transform, i.e.

$$u_k^{(n)}(\beta, S_n) = \frac{\Delta^k\left(\dfrac{(n+\beta)^{k-2} S_n}{\Delta S_{n-1}}\right)}{\Delta^k\left(\dfrac{(n+\beta)^{k-2}}{\Delta S_{n-1}}\right)} = \frac{\sum_{j=0}^{k} (-)^j \binom{k}{j} (\beta+n+j)^{k-2} \dfrac{S_{n+j}}{\Delta S_{n+j-1}}}{\sum_{j=0}^{k} (-)^j \binom{k}{j} (\beta+n+j)^{k-2} \dfrac{1}{\Delta S_{n+j-1}}}. \tag{16}$$

In arriving at the last step we have used the properties of the Δ-operator, i.e.

$$\Delta^k f(n) = \sum_{j=0}^{k} (-)^j \binom{k}{j} f(n+j).$$

For an alternating series Levin suggested $\omega_n = \Delta S_{n-1}$, and this gives the Levin t-transform

$$t_k^{(n)}(\beta, S_n) = \frac{\sum_{j=0}^{k} (-)^j \binom{k}{j} (\beta+n+j)^{k-1} \dfrac{S_{n+j}}{\Delta S_{n+j-1}}}{\sum_{j=0}^{k} (-)^j \binom{k}{j} (\beta+n+j)^{k-1} \dfrac{1}{\Delta S_{n+j-1}}}. \tag{17}$$

Smith and Ford [8] suggested $\omega_n = \Delta S_n$ for an alternating series. However, with this choice of ω_n, the transform would need one more term for a given order. In subsequent investigations we shall consider $n = 1$ and denote the u and t-transforms as $u_k(\beta)$ and $t_k(\beta)$.

A new class of sequence transforms is obtained [4] by replacing $(n+\beta)^i$ in Eq. (10) by $(n+\beta)_{(i)}$, i.e.

$$\frac{(S - S_n)(n+\beta)_{(k-1)}}{\omega_n} = \sum_{i=0}^{k-1} c_i (n+\beta)_{(k-i-1)}, \tag{18}$$

where $z_{(\nu)}$ is the Pochhammer symbol defined as

$$z(\nu) = \frac{\Gamma(z+\nu)}{\Gamma(z)} = z(z+1)\ldots(z+\nu-1). \tag{19}$$

With the choice $\omega_n = (\beta+n)\ \Delta S_{n-1}$ and $\omega_n = \Delta S_{n-1}$ one gets the y and τ-transforms respectively and they are given by

$$y_k^{(n)}(\beta,S_n) = \frac{\sum_{j=0}^{k}(-)^j\binom{k}{j}(\beta+n+j)_{(k-2)}\frac{S_{n+j}}{\Delta S_{n+j-1}}}{\sum_{j=0}^{k}(-)^j\binom{k}{j}(\beta+n+j)_{(k-2)}\frac{1}{\Delta S_{n+j-1}}}, \tag{20}$$

$$\tau_k^{(n)}(\beta,S_n) = \frac{\sum_{j=0}^{k}(-)^j\binom{k}{j}(\beta+n+j)_{(k-1)}\frac{S_{n+j}}{\Delta S_{n+j-1}}}{\sum_{j=0}^{k}(-)^j\binom{k}{j}(\beta+n+j)_{(k-1)}\frac{1}{\Delta S_{n+j-1}}}. \tag{21}$$

Here also we shall use $n=1$ and denote the corresponding transforms as $y_k(\beta)$ and $\tau_k(\beta)$ respectively.

All these transforms can be compactly written as

$$L_k(\beta,\delta) = \frac{\sum_{j=0}^{k}(-)^j\binom{k}{j}(\beta+j+1)^{k-\delta}\frac{S_{j+1}}{\Delta S_j}}{\sum_{j=0}^{k}(-)^j\binom{k}{j}(\beta+j+1)^{k-\delta}\frac{1}{\Delta S_j}}, \tag{22}$$

so that $L_k(\beta,2)=u_k(\beta)$ and $L_k(\beta,1)=t_k(\beta)$ and

$$W_k(\beta,\delta) = \frac{\sum_{j=0}^{k}(-)^j\binom{k}{j}(\beta+j+1)_{(k-\delta)}\frac{S_{j+1}}{\Delta S_j}}{\sum_{j=0}^{k}(-)^j\binom{k}{j}(\beta+j+1)_{(k-\delta)}\frac{1}{\Delta S_j}}, \tag{23}$$

where $W_k(\beta,2)=y_k(\beta)$ and $W_k(\beta,1)=\tau_k(\beta)$.

We can recast the transforms in the form

$$(L,W)_k(\beta,\delta) = \frac{\sum_{j=0}^{k}\nu_{kj}\frac{S_{j+1}}{\Delta S_j}}{\sum_{j=0}^{k}\nu_{kj}\frac{1}{\Delta S_j}} = \frac{\sum_{j=0}^{k}z^j\sum_{i=0}^{k}\omega_{ki}a_{j-i}}{\sum_{j=0}^{k}\omega_{kj}z^j},$$

where

$$\nu_{kj} = \begin{cases}(-)^j\binom{k}{j}(\beta+j+1)^{k-\delta} & \text{for the } L\text{-transform},\\ (-)^j\binom{k}{j}(\beta+j+1)_{(k-\delta)} & \text{for the } W\text{-transform},\end{cases}$$

and

$$\omega_{kj} = \begin{cases}(-)^j\binom{k}{j}\dfrac{(\beta+k-j+1)^{k-\delta}}{a_{k-j}} & \text{for the } L\text{-transform},\\ (-)^j\binom{k}{j}\dfrac{(\beta+k-j+1)_{(k-\delta)}}{a_{k-j}} & \text{for the } W\text{-transform}.\end{cases} \tag{24}$$

Thus L_k and W_k represent a set of rational functions, which are obtained from $(k+1)$ terms of the original sequence $\{S_n\}$ and are approximants for the function $f(z)$.

It can be easily demonstrated that

$$f(z) - T_k^{(n)}(\beta, \delta) = O(z^{n+k}), \tag{25}$$

where $T_k^{(n)}(\beta, \delta)$ stands for either L or W-transform.

Table 2
Magnitude of the relative error for different approximants at some speciifed points. For a particular value of x the first row gives the relative error for $\beta = 0$ and the second row gives that for $\beta = 1$.

Function	x	[2,2]	Transform used			
			u_4	t_4	y_4	τ_4
e^x	−0.5	4.40×10^{-5}	8.44×10^{-7}	2.31×10^{-7}	9.01×10^{-8}	2.62×10^{-9}
			5.18×10^{-6}	8.98×10^{-7}	8.72×10^{-6}	3.42×10^{-6}
	0.5	4.40×10^{-5}	9.03×10^{-7}	1.21×10^{-7}	1.21×10^{-7}	2.97×10^{-9}
			1.03×10^{-5}	1.61×10^{-6}	1.78×10^{-5}	6.38×10^{-6}
	−1.0	1.47×10^{-3}	5.32×10^{-5}	1.89×10^{-5}	1.02×10^{-5}	6.45×10^{-7}
			1.23×10^{-4}	2.08×10^{-5}	1.99×10^{-4}	8.10×10^{-5}
	1.0	1.47×10^{-3}	6.06×10^{-5}	4.43×10^{-6}	1.83×10^{-5}	8.31×10^{-7}
			4.78×10^{-4}	6.96×10^{-5}	8.21×10^{-4}	2.82×10^{-4}
$\log(1+x)$	−0.8	2.75×10^{-2}	7.52×10^{-3}	3.42×10^{-3}	4.70×10^{-3}	1.37×10^{-3}
			3.35×10^{-3}	1.45×10^{-3}	2.17×10^{-3}	6.26×10^{-4}
	1.0	1.21×10^{-3}	6.95×10^{-6}	4.68×10^{-6}	2.03×10^{-5}	3.66×10^{-6}
			3.66×10^{-6}	9.16×10^{-7}	6.95×10^{-6}	1.10×10^{-6}
	2.0	7.01×10^{-3}	7.44×10^{-5}	6.37×10^{-5}	2.37×10^{-4}	7.10×10^{-5}
			6.54×10^{-6}	2.26×10^{-5}	1.52×10^{-4}	3.82×10^{-5}
$(\sin x)/x$	1.0	2.70×10^{-8}	1.23×10^{-9}	1.48×10^{-10}	6.14×10^{-10}	1.18×10^{-10}
			6.96×10^{-9}	1.20×10^{-9}	1.15×10^{-8}	4.40×10^{-9}
	2.0	4.32×10^{-5}	2.27×10^{-6}	4.10×10^{-7}	4.63×10^{-7}	1.69×10^{-7}
			7.40×10^{-6}	1.17×10^{-6}	1.30×10^{-5}	4.86×10^{-6}
	4.0	5.51×10^{-2}	2.84×10^{-3}	1.51×10^{-3}	4.67×10^{-3}	9.86×10^{-4}
			3.22×10^{-3}	8.50×10^{-4}	2.25×10^{-3}	1.51×10^{-3}
$\phi(x,1.1,0)$	−0.9	8.87×10^{-4}	6.78×10^{-6}	2.77×10^{-6}	1.22×10^{-5}	1.79×10^{-6}
			4.06×10^{-6}	5.71×10^{-7}	1.18×10^{-6}	4.78×10^{-7}
	1.0	0.75	1.79×10^{-4}	0.69	1.18×10^{-3}	0.68
			3.48×10^{-3}	0.68	8.98×10^{-3}	0.66
	0.5	1.14×10^{-3}	7.42×10^{-5}	2.88×10^{-5}	3.13×10^{-5}	4.75×10^{-6}
			1.69×10^{-5}	6.04×10^{-6}	5.61×10^{-6}	1.61×10^{-6}
$\int_0^\infty \frac{e^{-t}dt}{1+xt}$	1.0	3.19×10^{-2}	1.65×10^{-3}	3.49×10^{-4}	7.89×10^{-5}	4.33×10^{-4}
			3.36×10^{-4}	2.98×10^{-4}	5.66×10^{-4}	9.00×10^{-5}
	2.0	0.11	1.04×10^{-4}	3.21×10^{-3}	3.93×10^{-3}	5.37×10^{-4}
			2.72×10^{-3}	4.96×10^{-4}	2.09×10^{-3}	1.42×10^{-3}
${}_2F_0(1,1,-1/x)$	1.0	3.19×10^{-2}	1.65×10^{-3}	3.49×10^{-4}	7.89×10^{-4}	4.33×10^{-4}
			3.34×10^{-4}	2.98×10^{-4}	5.66×10^{-4}	9.00×10^{-5}
	2.0	6.39×10^{-3}	6.02×10^{-4}	2.80×10^{-4}	2.82×10^{-4}	5.87×10^{-5}
			1.20×10^{-4}	3.33×10^{-5}	2.18×10^{-5}	4.08×10^{-5}
	4.0	8.53×10^{-4}	9.36×10^{-5}	1.05×10^{-4}	5.91×10^{-5}	2.09×10^{-5}
			3.69×10^{-5}	2.18×10^{-5}	2.27×10^{-5}	1.28×10^{-6}

If we use the symbol $\Pi_n(z)$ to denote any polynomial z of degree n, then

$$S_n = \Pi_n(z). \tag{26}$$

Assuming that $a_i \neq 0$ for all $i > 0$, it follows that

$$\Delta^i S_n = z^n \Pi_{i-1}(z). \tag{27}$$

For convenience let us write

$$\phi_n = \begin{cases} \dfrac{(n+\beta)^{k-\delta}}{a_n z^{n-1}} & \text{for } L\text{-transform}, \\ \dfrac{(n+\beta)_{(k-\delta)}}{a_n z^{n-1}} & \text{for } W\text{-transform}, \end{cases} \tag{28}$$

so we can write

$$\Delta^i \phi_n = \frac{\Pi_i(z)}{z^{n+i-1}}, \tag{29}$$

so the kth order transform may be written as

$$T_k^{(n)}(\beta,\delta) = \frac{\Delta^k(\phi_n S_n)}{\Delta^k \phi_n}. \tag{30}$$

Now

$$\Delta^k(\phi_n S_n) = \sum_{j=0}^{k-1} (-)^j \binom{k}{j} \Delta^{k-j} S_{n+j} \Delta^j \phi_n + S_{n+k} \Delta^k \phi_n. \tag{31}$$

Hence

$$T_k^{(n)}(\beta,\delta) = S_{n+k} + z^{n+k} \frac{\Pi_{k-1}(z)}{\Pi_k(z)}, \tag{32}$$

which proves that $n+k$ terms of the power series expansion of $T_k^{(n)}(\beta,\delta)$ will again agree with those of the original series for any value of β, δ. The $[m,n]$ Padé approximant is built in such a way that $m+n+1$ terms of the series for the approximant will agree with its actual series. For a Levin-like transform $T_k^{(n)}(\beta,\delta)$, $n+k$ terms of the series expansion of the approximant will definitely agree for any value of β and δ and depending on the values of β and δ more terms may agree with the actual series.

Table 3
Number of terms of the series expansion of different approximants of order 4 that agree with the actual series. Five terms have been used to build each of the approximants.

Function	$u_4(0)$	$u_4(1)$	$t_4(0)$	$t_4(1)$	$y_4(0)$	$y_4(1)$	$\tau_4(0)$	$\tau_4(1)$
e^x	5	5	6	5	7	5	8	5
$\log(1+x)$	5	6	5	6	6	7	7	8
$(\sin x)/x$	5	5	5	5	5	5	5	5
$\phi(x,1.1,0)$	5	5	5	5	5	5	5	5
$\int_0^\infty \frac{e^{-t}dt}{1+xt}$	6	6	5	5	6	6	5	5
${}_2F_0(1,1,-1/x)$	6	6	5	5	6	6	5	5

In a recent paper [12] we compared the performance of the approximants obtained form $u_k(0)$ with the Padé approximants and found that for a given number of terms of the original series the $u_k(0)$-approximants are, in most cases, better representations of the corresponding functions. A comparison of the two methods made on a divergent perturbation series expansion for the excluded-volume effect in the theory of polymer solutions extends support to this surmise [14]. Thus it seems natural to make a comparative study of the different Weniger approximants along with the Padé approximants and we undertake it in the subsequent section.

4. A comparative study of Padé approximants and Weniger approximants on some test series

In this section we make a comparative study of Padé approximants and Weniger approximants calculated from the series expansion of some known functions. Table 1 is a list of functions and the approximants of order 4 which require five terms of the original series. The Padé approximants which also require the same number of terms are also given in the table. The numerical efficiency of these approximations in relation to Padé approximants at some specified points for these functions are given in Table 2. It is evident from the table that for these representative convergent, divergent and asymptotic series, the Weniger approximants are significantly better. The comparison of the actual functions with their Padé approximants and the other approximants is shown in Figs. 1–6. Instead of plotting the actual functions and the approximants, we plot the logarithm (to the base 10) of the absolute value of the difference between the function and the approximant against the independent variable and this makes the comparison much more transparent. The number of terms of the series expansion of the approximants which agree with those of the original series for the different test functions chosen is shown in Table 3.

The first example is on approximating e^x in the interval $(-\infty,\infty)$. It is well known that in binary arithmetic the problem is reduced to one of approximating e^x in the finite interval $(-\ln 2, \ln 2)$ or approximately, $(-0.7, 0.7)$. Fig. 1 shows the log-plot for the different approximants of order four with $\beta = 0$. It is evident from the figure that the Weniger approximants are distinctly better than the [2,2] Padé approximant. The relative error at $x = -0.5$ with τ_4 is 2.6×10^{-9}, whereas that with the [2,2] Padé approximant is only 4.4×10^{-5}. It may be remarked that the series for e^x is a monotone sequence for positive x and is an alternating one for negative x. Though the t-transform is built for an alternating series, it works quite well for positive and negative x. It is seen from the figure that the curves for y_4 and t_4 cross at nearly $x = 0.5$. Thus, if one uses a transform on a sequence of numbers, at $x = 0.2$ the y_4-transform will work better then t_4-transform; but the reverse will be the case if one uses the sequence of numbers at $x = 1.0$. This type of crossing is common for almost all the functions for any order of the approximants. Thus for a study on sequences of numbers it is difficult to conclude which transforms work better for the function. From Fig. 1 and Table 2 one can conclude that $\tau_4(0)$ is the best representation for e^x. It is seen from Table 3 that 8 terms of the series expansion for $\tau_4(0)$ agree with those of the original series, though only five terms have been used to build the approximant.

The next series we consider is that for $\ln(1 + x)$ which can be represented as a Stieltjes integral, i.e.

$$\ln(1+x) = x\int_0^1 \frac{\mathrm{d}t}{1+xt} = x\sum_{m=0}^{\infty} \frac{(-x)^m}{m+1}. \tag{33}$$

The power series converges absolutely for $|x| < 1$ and for $x > 1$ the series diverges. Moreover, for positive x the series is alternating and for negative x the series is monotone. This series gives the Madelung sum for an one-dimensional ionic lattice and also occurs if correlation effects in atoms are treated by perturbation theory [15]. It may be remembered here that many perturbation series expansions are Stieltjes series. Incidentally, Grotendorst [16] lists the approximants for the above function obtained from a number of nonlinear transforms.

Fig. 2 shows the log-plot for the difference between the approximants of order 4 and the function. Here also the Weniger approximants better reproduce the function than the [2,2] approximant over the entire region – monotone, alternating and divergent. Of the Levin-like transforms the τ-transform seems to work better, if one ignores the "spikes" in the curves. Table 2 also supports this fact as the relative error for τ_4 is less than that for other transforms for the values shown in Table 2. The relative error for $\tau_4(1)$ at $x = 1$ is 1.1×10^{-6} whereas that for [2,2] Padé approximant is 1.3×10^{-3}. From the Table 3 it may be seen that eight terms of the series expansion for $\tau_4(1)$ agrees with the original series, whereas only five terms are used to build the transform. A look at Fig. 3 reveals that the curves for different approximants cross each other at a number of points. If one uses the different transforms on the sequence of numbers at $x = -0.5$ one comes to the conclusion that $\tau_4(0)$ is the best transform. But from an identical performance on the sequences of numbers for $x = 0.5$ and $x = 1.4$, it will be concluded that t_4 and τ_4 are respectively the best transforms for the function. This characteristic crossing of the curves for the different approximants is common for any order for any function. Thus conclusions from a study on sequences of numbers should be taken with great caution. It may be remarked here that the spikes represent points where the function is exactly reproduced by the corresponding transform. We shall discuss this point in the final section.

Fig. 3 shows a similar plot $(\sin x)/x$ which is an oscillating function and consequently has a number of zeros. For this function also the Weniger approximants are better than the [2,2] Padé approximant. The performance of the different Weniger approximants is more or less the same except at the spikes, where the function is exactly reproduced. From Table 3 it is seen that only five terms of the series for the different approximants agree with the actual series. Table 2 gives the relative error for the function with different approximants at some specified points. Here also the Weniger approximants cross each other at a number of points. Consequently it is difficult to come to any conclusion about the overall performance of a particular transform from a study on sequences of numbers.

Series 4 is a special case of the Lerch transcendent Φ-function defined by

$$\Phi(x,s,0) = \sum_{n=1}^{\infty} \frac{x^n}{n^s} \tag{34}$$

and for $x = 1$ and -1 it represents respectively $\zeta(s)$ and $\eta(s)$. Fig. 4 shows the log-plot for different Weniger approximants along with [2,2] Padé approximant and the Weniger approximants are found to be superior to the Padé approximant. From Table 2 it is seen that at $x = 1$ the relative errors for [2,2] Padé approximant, t_4 and τ_4

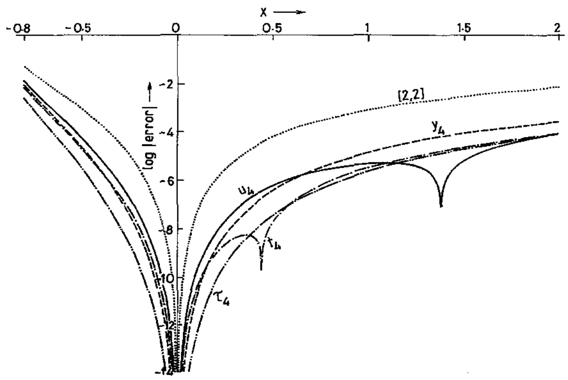

Fig. 2. Plot of the logarithm of the absolute error for $\log(1 + x)$ against x using different approximants of order 4 with $\beta = 0$.

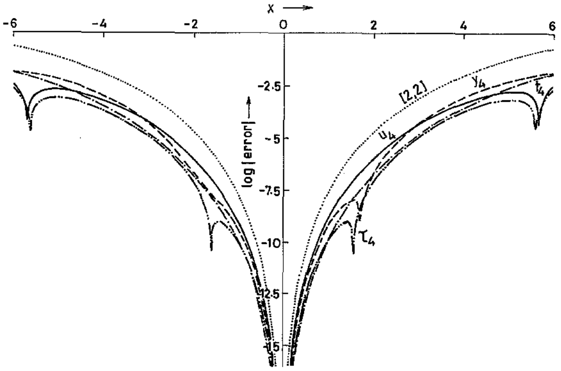

Fig. 3. Plot of the logarithm of the absolute error for sin x/x against x using different approximants of order 4 with $\beta = 0$.

are comparatively large. Apart from the region near $x = 1$, all the Weniger transforms reproduce the function fairly well. The u_4 and y_4 seem to reproduce the function better over the entire region. As can be seen from Table 3, six terms of the series expansion of u_4 and y_4 agree with that of the original series, unlike t_4 and τ_4 for which only five terms agree. For this function also there are crossings of the curves for different approximants. There are spikes or points of exact coincidence in the curves for some of the approximants.

Series 5 is a divergent Stieltjes series and is obtained by expanding as an infinite power series in z the function

$$f(z) = \int_A^B \frac{\rho(t)}{1+zt} \mathrm{d}t, \tag{35}$$

where $\rho(t)$, t are real and z is complex and each of the limits A and B $(B > A)$ may be finite or infinite. Taking $\rho(t) = \mathrm{e}^{-t}$ and the range of integration $(0,\infty)$ and expanding $(1 + zt)^{-1}$ one gets the famous Euler series given in Table 5. Even though $f(z)$ is finite and can be computed by numerical integration, the series for $f(z)$ is divergent. This series is a good mathematical model for the kind of divergence which occurs in the perturbation series for the quartic anharmonic oscillator. In Table 1 we give the different approximants which reproduce the

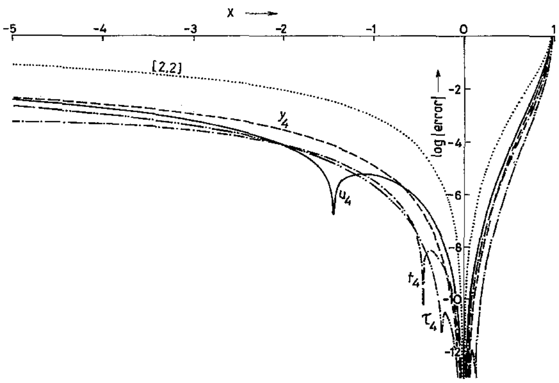

Fig. 4. Plot of the logarithm of the absolute error for $\phi(x,1.1,0)$ against x using different approximants of order 4 with $\beta = 0$.

function over a wide range of the argument. A close relative of the Euler integral is the so-called exponential integral

$$E_1(z) = \int_z^\infty \frac{e^{-x}}{x} dx. \tag{36}$$

Using the transformation $x = t + z$ one gets

$$z e^z E_1(z) = \int_0^\infty \frac{e^{-t} dt}{1 + tz} = \sum_{m=0}^\infty (-)^m m! z^{-m} = {}_2F_0\left(1,1,-\frac{1}{z}\right). \tag{37}$$

This is the asymptotic hypergeometric series we consider as the last example (series 6 in Table 1). The radius of convergence of this hypergeometric series is zero and diverges for any values of z. To build an approximant for an asymptotic series we replace x by $1/y$ so that it is a power series in y. We then build the approximant in y and replace y by $1/x$. Some of the Weniger approximants of order 4 are given in Table 1. Figs. 5 and 6 show the log-plot for the Euler and the asymptotic series. In both cases the Weniger approximants are distinctly better than [2,2] Padé approximants. Also, six terms of the series expansion for u_4 and y_4 agree with those of the original series. From Table 2 it can be seen that the relative error for the Euler function at $x = 2$ is 0.11, whereas those for the Levin approximants vary between 10^{-3} to 10^{-5}. The relative error for the asymptotic series decreases for larger values of x as expected. Due to crossing and the presence of spikes it is difficult to decide about the superiority of one Weniger transform over another, but it can definitely be concluded that they work much better than the Padé approximants with the same number of terms of the original series.

It is also worthwhile to investigate the performance of higher order Padé approximants with that of a given order of the Weniger transforms. We find, for example that for the functions e^x and $\log(1+x)$, the Padé approximants [3,4], [5,6] and [7,8] are comparable with τ_4, τ_6 and τ_8, respectively. For the divergent Euler series (number 5 in Table 1) the approximant [15,15] is better than τ_6 in the range (0,1), the two are comparable in the range (1,3), but τ_6 is significantly better than [15,15] for $x > 3$. We note again that the transform τ_k needs $k+1$ terms of a series whereas the Padé approximant $[m,n]$ needs $m+n+1$ terms.

5. Discussion on the choice of the parameter β

The Weniger transforms contain a free parameter β. It has also been demonstrated in Section 3 that

$$f(z) - T_k^{(n)}(\beta,\delta) \equiv O(z^{n+k}) \tag{38}$$

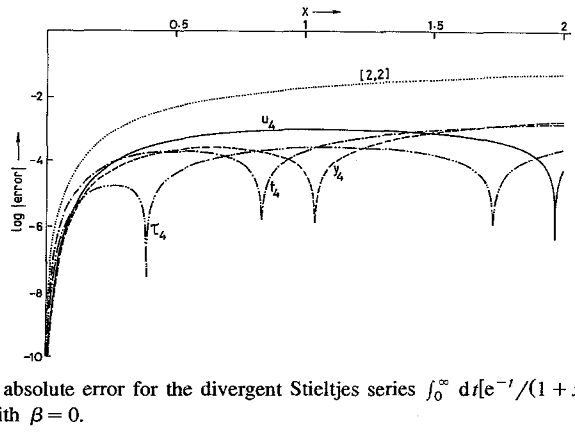

Fig. 5. Plot of the logarithm of the absolute error for the divergent Stieltjes series $\int_0^\infty dt[e^{-t}/(1+xt)] = \sum_0^\infty (-x)^m m!$ against x using different approximants of order 4 with $\beta = 0$.

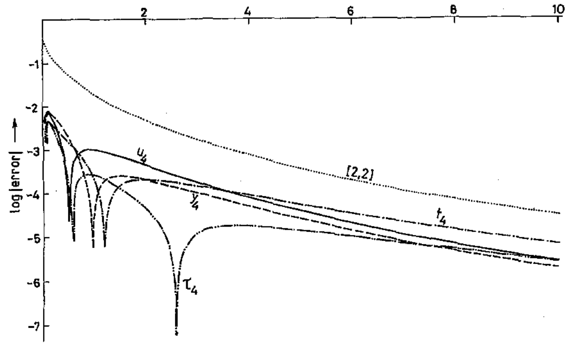

Fig. 6. Plot of the logarithm of the absolute error for the asymptotic hypergeometric series ${}_2F_0(1,1,-1/x)$ against x using different approximants of order 4 with $\beta = 0$.

for any value of β, where $T_k^{(n)}(\beta,\delta)$ represent the L or W-transform and δ is chosen to be either 1 or 2. With $\delta = 1$ one obtains either the t or τ transforms and with $\delta = 2$ the u and y transforms are obtained. Any of the approximants discussed in Section 3 for a function $f(x)$ at a certain value of x approximates the function at that point. Table 4 presents the different approximants for the function $\log(1+x)$ at $x = 1$, i.e. for log 2. These approximants with β ranging from -1 to 1 approximate log 2 to an accuracy of 0.02%. This fact is clearly seen from Fig. 7, where we have plotted the difference between the approximants and log 2 against β. The question that naturally arises is whether we can choose a value of β for which the function is best approximated. The answer to this question is closely related to the spikes in the logarithmic plots (Figs. 1–6). The spikes in these plots arise at those values of x for which the approximants reproduce the function to the available precision corresponding to the values of β given in the plots. It can be easily understood that if we equate the function to $T_k^{(n)}$ for a particular value of x we obtain a polynomial equation for β, i.e.

$$f(x)\,D(\beta,x) - N(\beta,x) = 0, \tag{39}$$

where $N(\beta,x)$ and $D(\beta,x)$ are the numerator and denominator of the approximant. The roots of these equations represent values of β for which the approximant approximates the function to the precision available. For t_k or τ_k we get a polynomial equation of order $k-1$ and for y_k or u_k the polynomial equations are of order $k-2$. Thus for t_2 or τ_2 we have a linear equation in β and for a real sequence the root of the equation is

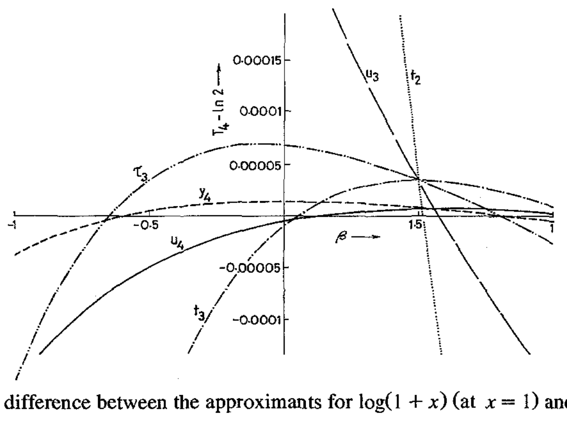

Fig. 7. Plot of the difference between the approximants for $\log(1+x)$ (at $x = 1$) and log 2 against β.

Table 5
Different approximants for $\log(1+x)$ at $x=1$

Transform	Approximant
$t_2(\beta)$ or $\tau_2(\beta)$	$\frac{25+11\beta}{4(9+4\beta)}$
$t_3(\beta)$	$\frac{707+466\beta+83\beta^2}{12(85+56\beta+10\beta^2)}$
$\tau_3(\beta)$	$\frac{940+549\beta+83\beta^2}{12(113+66\beta+10\beta^2)}$
$u_3(\beta)$ $y_3(\beta)$	$\frac{233+83\beta}{24(14+5\beta)}$
$u_4(\beta)$	$\frac{4791+2662\beta+399\beta^2}{192(36+20\beta+3\beta^2)}$
$y_4(\beta)$	$\frac{6122+3061\beta+399\beta^2}{192(46+23\beta+3\beta^2)}$

always real. Similar is the situation for u_3 or y_3. For t_3, τ_3, u_4 and y_4 the equations for β are quadratic and for a real sequence either both roots are real or both of them are complex depending on the value of x and the sequence considered. The variation of β with x for the different test functions considered in Table 1 with different approximants are shown in Figs. 8–13. For t_2 or τ_2 and u_3 or y_3 the variation of β with x is smooth and for any value of x there is a value of β for which the function is reproduced to the available precision. For these approximants β decreases monotonically with x for the test functions 1,2 and 5 of Table 1 and the opposite is the case for other three test functions. For the approximants u_4 or y_4 and t_3 or τ_3 the equation for β is quadratic. Thus for a real sequence either both the roots are real or both of them are complex. For higher order approximants it is always likely that some of the roots are complex. This is evident from the Figs. 8–13. From Fig. 8 it may be seen that the two values of β, corresponding to t_3 or τ_3 and u_4 or y_4, approach each other as x decreases and, at a certain value of x, the two roots are coincident. For lower values of x the roots for β become complex and are consequently not shown in the figure. A similar situation arises for other

Table 5
Table for the values of β for which the functions $\log(1+x)$ and $\int_0^\infty (\mathrm{d}t e^{-t}/1+xt)$ are reproduced to the available precision at different values of x with the transform $\tau_2(\beta)$

x	β for $\log(1+x)$	β for $\int_0^\infty \frac{e^{-t}dt}{1+xt}$
0.5	0.7008248699351658	2.732760300916694
1.0	0.5168674366153715	1.418804735230054
2.0	0.2926715667402574	0.6400859679621395
3.0	0.1556334097334717	0.331638394423988
4.0	0.06051251907255196	0.1576745389821985
5.0	−0.01060629140089863	0.04297590304242866
6.0	−0.06644882881331578	−0.03969723921820527
7.0	−0.1118485927370422	−0.102832538007318
8.0	−0.1497101220985562	−0.1530427329933659
9.0	−0.1819856149772059	−0.1941911259255686
10.0	−0.2098952530131156	−0.2287022298920801

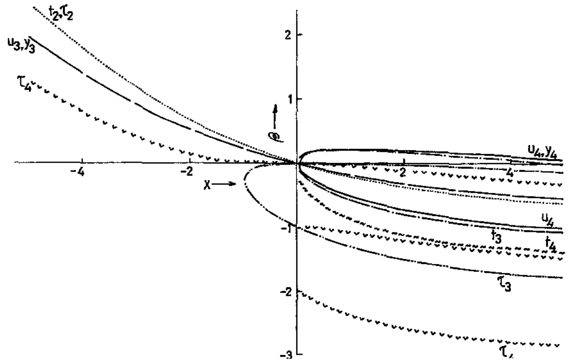

Fig. 8. Plot of β against x for approximants of different orders for the function e^x.

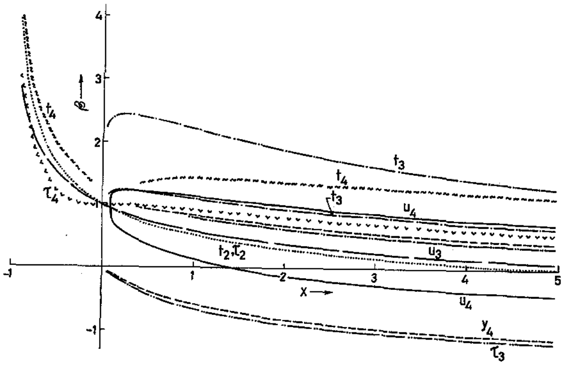

Fig. 9. Plot of β against x for approximants of different orders for the function $\log(1+x)$.

functions also like $\log(1+x)$ or $\phi(x,1.1,0)$ as may be seen in Figs. 9 and 11. We would like to remark here that the variation of β with x may or may not be similar for different orders. For example, in the case of the test functions 4 and 6, β increases with x in the case of t_2 or u_3 but it decreases with x in case of t_3 and u_4 as may be seen from Figs. 11 and 13. On the other hand, the variation of β with x in case of series number 5 is almost identical for the different orders. For this test function the β values for x in the range 0–0.1 are not

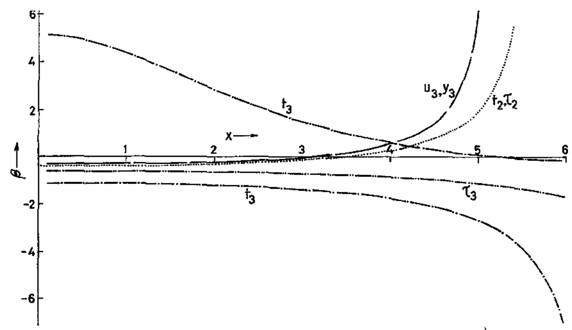

Fig. 10. Plot of β against x for approximants of different orders for the function $(\sin x)/x$.

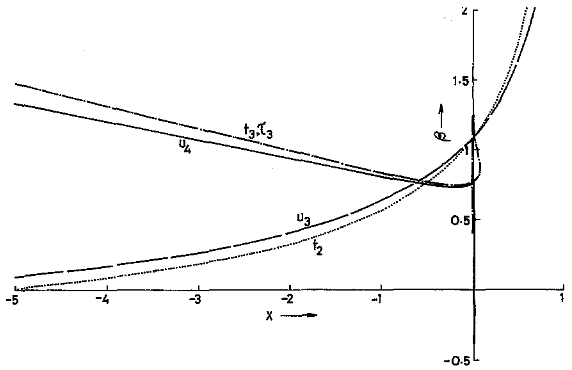

Fig. 11. Plot of β against x for approximants of different orders for $\phi(x,1.1,0)$.

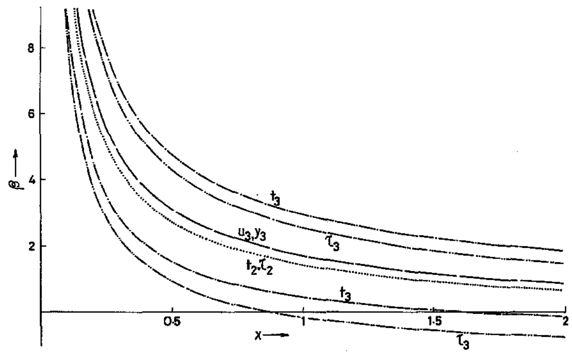

Fig. 12. Plot of β against x for approximants of different orders for the Stieltjes integral $\int_0^\infty \mathrm{d}t[\mathrm{e}^{-t}/(1+xt)]$.

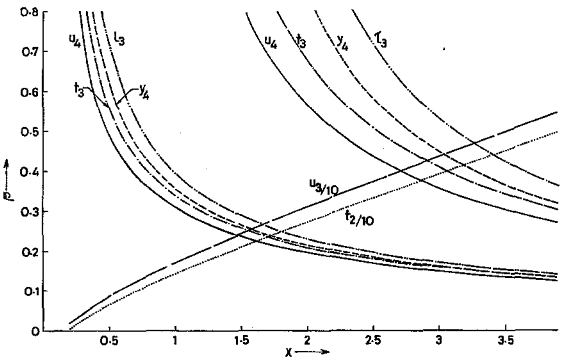

Fig. 13. Plot of β against x for approximants of different orders for the asymptotic hypergeometric series ${}_2F_0(1,1,-1/x)$.

shown in the figure. In this range β starts with a small value at $x=0$, increases to a very high value of the order of 300 and then decreases monotonically. This part of the variation is not shown in the figure.

We also note that with a complex value of β the approximant becomes complex. Yet the real part of this approximant is a fair representation of the real function. For example, the real part of the approximant t_3 (β) for the function e^x with $\beta = 0.2290251 - 0.510364\ i$ reproduces the function e^x up to four decimal places for

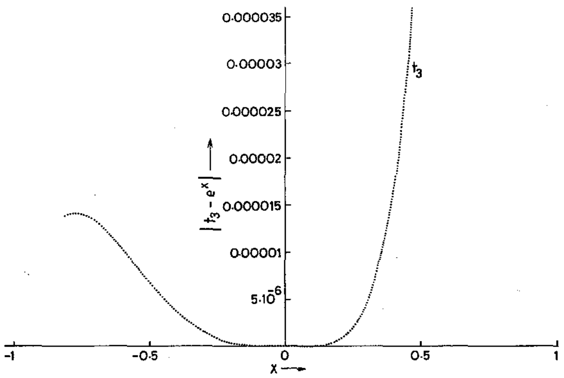

Fig. 14. Plot of the absolute value of the difference between the real part of t_3 and e^x against x with $\beta = 0.2290251 - 0.510364\,i$.

values x in the range -1 to 1 as may be seen from Fig. 14. The above value of β is chosen in such a way that $t_3(\beta)$ exactly reproduces the function at $x = -1$. It may be noted at this point that if we use a complex value for β and take the real part of the approximant as a representation of the function then the order of the polynomials of both the numerator and denominator of the approximant is doubled.

In order to demonstrate now effective an appropriate choice of β can be, we first show in Fig. 15 the Weniger approximants for tan x with $\beta = 0$. The [2,2] Padé approximant clearly outperforms the rest. Next using $\tau_2(x,\beta)$, which uses only three terms of the series for tan x, we solve for β from $\tau_2(x,\beta) = \tan x$. This equation, being transcendental, can be used to get β as a power series in x by (i) expanding it as a Taylor series about $x = 0$ and (ii) fitting β as a least squares polynomial. If these expressions for β are substituted back in $\tau_2(x,\beta)$, the result is a higher order approximant whose deviation from tan x is shown in Fig. 16. In this figure the curves labelled σ_i correspond to τ_2 with β replaced by i terms of its Taylor series expansion. Similarly the ones labelled λ_i correspond to least square fittings of β with i terms. It is seen that the third order least squares fit reproduces tan x with an accuracy of 10^{-10} over the range 0–1.2. Thus it is possible to fit known functions with low order transforms and an appropriate choice of β with a much greater accuracy than a higher order transform with a fixed β (e.g., $\beta = 0$).

6. Conclusions and discussions

Here we have extended our previous work on rational approximants generated by the u-transform to include the generalizations of the Levin transforms made by Weniger. The effectiveness of the approximants generated

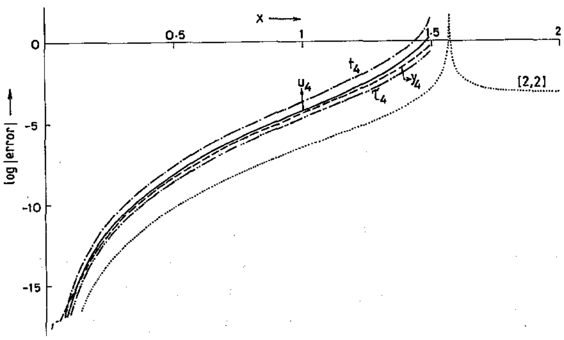

Fig. 15. Plot of the logarithm of the absolute error for tan x against x using different approximants of order 4 with $\beta = 0$.

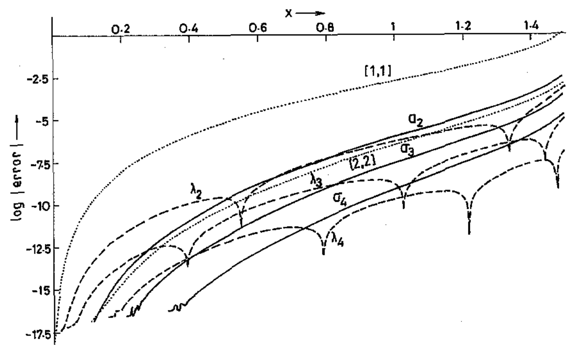

Fig. 16. Plot of the logarithm of the absolute error for tan x against x for τ_2 (x,β) with λ_i and σ_i corresponding to series expansions and least square fits respectively for β with i terms.

by the Weniger transforms are compared with both Padé approximants as well as the u-approximants studied previously.

We conclude that the y and τ-transforms, which are, respectively, the generalizations of the u and t-transforms, are not only more effective than the Padé approximant, but mostly produce better approximants than u and t. The results obtained also bring into question the wisdom of judging the effectiveness of nonlinear transforms by applying them only on sequences of numbers. As we have shown, even for a given function the effectiveness of the approximants varies with the argument. Thus different conclusions can be drawn from two different sequences of numbers which really represent the values of one function at two different points.

The study of the approximants also shows at a glance the effectiveness of the transforms on monotone, alternating and divergent sequences simply by applying them on the series expansion of, say, $\log(1+x)$. As can be seen from Fig. 2, the ranges $-1<x<0$, $0<x<1$ and $x>1$ correspond, respectively, to monotone, alternating and divergent series. This study of the approximants, therefore, brings into focus in a unified manner a large number of properties of nonlinear transforms which generate them.

We also explore the freedom of choosing the parameter β which appears in the Weniger transforms. We show that by choosing β appropriately it is possible to use a relatively low order transform to represent the function to the accuracy of the available precision over a whole range of the argument. Even if β is chosen to be a complex number, the real part of the approximant reproduces the function fairly well. Finally, even for a function like tan x which has both zeros and poles for which the Weniger transforms are not so effective in comparison with the Padé approximant, a choice of β as an appropriate power series enables these transforms to far exceed the performance of Padé approximants.

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