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Vector Models in $\mathcal{PT}$ Quantum Mechanics

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Abstract We present two examples of non-Hermitian Hamiltonians which consist of an unperturbed part plus a perturbation that behaves like a vector, in the framework of $\mathcal{PT}$ quantum mechanics. The first example is a generalization of the recent work by Bender and Kalveks, wherein the E2 algebra was examined; here we consider the E3 algebra representing a particle on a sphere, and identify the critical value of coupling constant which marks the transition from real to imaginary eigenvalues. Next we analyze a model with SO(3) symmetry, and in the process extend the application of the Wigner-Eckart theorem to a non-Hermitian setting.

Keywords Non-Hermitian quantum mechanics · PT quantum mechanics · Wigner-Eckart theorem

1 Introduction

There are many situations in quantum mechanics wherein the Hamiltonian under consideration can be written as

$$H = H_0 + H_1 \tag{1}$$

where H_0 is the unperturbed part and commutes with the generators T_i of symmetry group G:

$$[H_0, T_i] = 0 \tag{2}$$

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and H_1 can be treated like a perturbation and behaves like a vector under G. We wish to generalize this situation in the context of $\mathcal{PT}$ quantum mechanics [1, 2], where the assumption that operators such as the Hamiltonian are Hermitian is relaxed, and replaced by other requirements, notably that the Hamiltonian commutes with the parity ($\mathcal{P}$) and time-reversal ($\mathcal{T}$) operators.

Interest in non-Hermitian quantum mechanics continues to grow [3], and recently a number of experiments have observed the so-called $\mathcal{PT}$ phase transition, where the eigenvalues of a $\mathcal{PT}$ Hamiltonian make a transition from being complex to real once a critical value of a coupling constant is reached [4–6]. Thus it is relevant to seek new $\mathcal{PT}$-counterparts to conventional Hamiltonians.

In this work we present two simple cases that can be described as non-Hermitian vector perturbation models where the Hamiltonian can be written as in Eq. (1); first we consider a particle confined to the surface of a sphere, where the Hamiltonian acts within an infinite dimensional Hilbert space, and next we consider a generic vector perturbation within a finite dimensional Hilbert space and determine the spectrum of eigenvalues using the Wigner-Eckart theorem. We find that for a range of parameters each of these models has a pure real spectrum. At critical values of the coupling the model undergoes $\mathcal{PT}$ transitions wherein the eigenvalues become complex.

2 E3 Algebra: Particle on a Sphere

We begin by generalizing the analysis presented in [7]. They considered the E2 algebra which consists of elements J, u, v such that

$$[J, u] = -iv, \qquad [J, v] = iu, \quad [u, v] = 0. \tag{3}$$

The Hamiltonian

$$h = J^2 + igu, \tag{4}$$

where $J = -i\partial/\partial\theta$, $u = \sin\theta$, $v = \cos\theta$ and g is a constant, represents a 2-dimensional quantum particle restricted to radius $r = 1$.

A generalization of this is the E3 algebra and restricting the particle to the surface of a sphere ($r = 1$). This is described by the Hamiltonian

$$h = L^2 + igu_z, \tag{5}$$

where L obeys

$$[L_i, L_j] = i\epsilon_{ijk}L_k \tag{6}$$

u is a vector operator whose components are given by

$$u_x = \sin\theta\cos\phi, \tag{7}$$

$$u_y = \sin\theta\sin\phi, \tag{8}$$

$$u_z = \cos\theta \tag{9}$$

and g is a constant. The remaining commutators are straightforward to calculate;

$$[L_i, u_j] = i\epsilon_{ijk}u_k, \qquad [u_i, u_j] = 0. \tag{10}$$

Following Bender and Kalveks we consider the case of even time reversal: for a wave function $\psi(\theta,\phi)$ the time reversal operator $\mathcal{T}$ is manifested as complex conjugation:

$$T\psi(\theta,\phi)=\psi^*(\theta,\phi) \tag{11}$$

hence $\mathcal{T}^2=1$. It is easy to verify the action of $\mathcal{T}$ on the elements of the algebra: $\mathcal{T}L_i\mathcal{T}=-L_i$ and $\mathcal{T}u_i\mathcal{T}=u_i$. The parity operator $\mathcal{P}$ takes ψ into the antipodal point:

$$P\psi(\theta,\phi)=\psi(\pi-\theta,\phi+\pi) \tag{12}$$

so $\mathcal{P}^2=1$; elements transform under parity as $\mathcal{P}L_i\mathcal{P}=L_i$ and $\mathcal{P}u_i\mathcal{P}=-u_i$. Note that the Hamiltonian h in Eq. (5) commutes with the combined operation $\mathcal{PT}$ but not with $\mathcal{P}$ or $\mathcal{T}$ individually. Now let us determine the eigenvalue spectrum of this Hamiltonian. We wish to solve

$$h\psi(\theta,\phi)=\lambda\psi(\theta,\phi) \tag{13}$$

and we try the general solution:

$$\psi(\theta,\phi)=f(\theta)e^{im\phi}. \tag{14}$$

For convenience we define $\eta=\cos\theta$; this simplifies the eigenvalue equation for f:

$$-\left(1-\eta^2\right)\frac{\partial^2 f}{\partial\eta^2}+2\eta\frac{\partial f}{\partial\eta}+\frac{m^2}{1-\eta^2}f+ig\eta f=\lambda f, \tag{15}$$

where m is a fixed integer. If we let

$$h_0=-\left(1-\eta^2\right)\frac{\partial^2 f}{\partial\eta^2}+2\eta\frac{\partial f}{\partial\eta}+\frac{m^2}{1-\eta^2}f \tag{16}$$

then the Hamiltonian we wish to solve is

$$h_0 f+ig\eta f=\lambda f. \tag{17}$$

We impose the boundary condition that the solution must be regular at $\eta=\pm1$.

Let us choose basis elements

$$|l\rangle\to N_l P_{l,|m|}(\eta), \tag{18}$$

where $l=|m|,|m|+1,\dots,$ $P_{l,|m|}$ are the associated Legendre polynomials, with conventional normalization factor

$$N_l=\sqrt{\frac{(2l+1)}{2}}\sqrt{\frac{(l-|m|)!}{(l+|m|)!}}. \tag{19}$$

The $P_{l,m}$'s satisfy

$$h_0 P_{l,|m|}(\eta)=l(l+1)P_{l,|m|}(\eta) \tag{20}$$

so the matrix of h_0 in this basis is diagonal. The matrix elements of the potential term, $ig\eta$, can easily be determined from the normalization and recursion relations of the $P_{l,m}$'s. By diagonalizing the truncated Hamiltonian matrix we can numerically obtain the eigenvalues of Eq. (17); see Fig. 1.

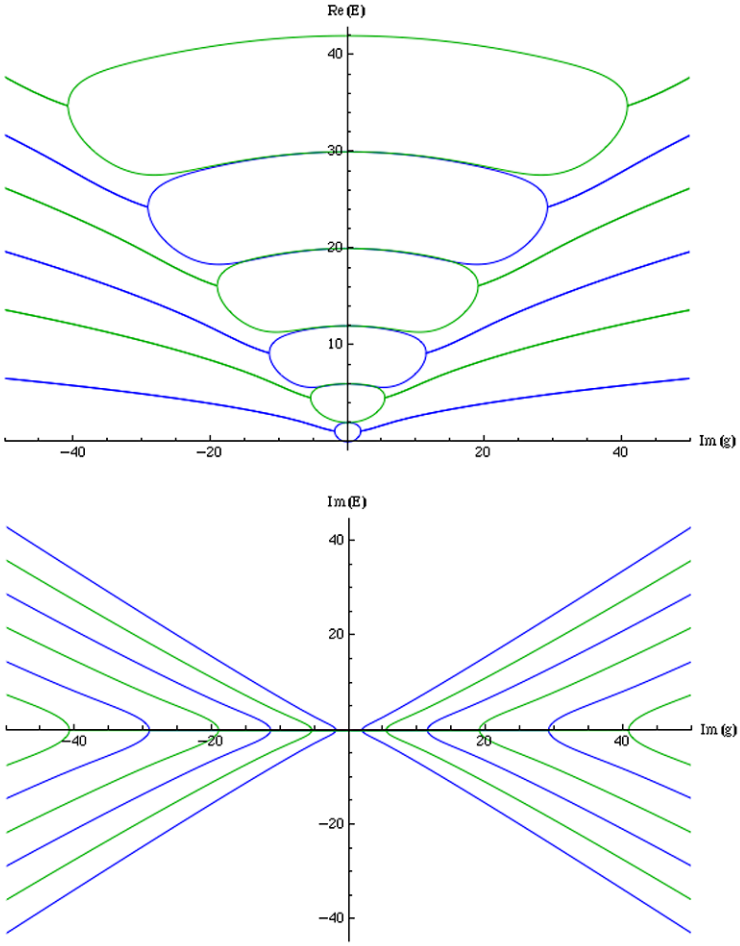

Fig. 1 Real and imaginary components of eigenvalues E for the Hamiltonian given by Eq. (17). The first six eigenvalues for $m = 0$ (*blue*) and $m = 1$ (*green*) are shown. Intercepts on the E axis are given by $\ell(\ell + 1)$ for $\ell = 0$ to 6. For the case of $m = 0$, we find that the spectrum is entirely real for $0 \leq g < 1.899$ at which point there is a transition to one pair of complex conjugate eigenvalues in the spectrum. At $g = 11.45$ there is a second transition, to two pairs of complex conjugate eigenvalues. Similarly for the case of $m = 1$, we find one complex conjugate eigenvalue pair at $g = 5.41$, and two pairs at $g = 19.04$. In these computations the Hamiltonian is truncated to a 100×100 matrix; we have verified that the relevant part of the spectrum is insensitive to the truncation

3 $\mathcal{PT}$ Vector Model in Finite-Dimensional Hilbert Space

$E3$ may also be regarded as a realization of the $\mathcal{PT}$ vector model with symmetry group $SO(3)$ and for which the Hilbert space is infinite dimensional. Now we wish to turn out attention to realizations of the $\mathcal{PT}$ vector model with finite dimensional Hilbert spaces. Let us write a simple, generic Hamiltonian $H = H_0 + H_I$ where

$$H_0 = L_x^2 + L_y^2 + L_z^2, \tag{21}$$

$$H_I = V_z \tag{22}$$

and V_z is the z component of a vector operator.

Our task is to obtain a matrix representation of the total Hamiltonian, solve for its eigenvalues and determine what value of the non-Hermitian perturbation cause the eigenvalues to become complex.

Naturally we choose to work with the angular momentum eigenstates $|\ell, m\rangle$; the action of H_0 on these states is well known, and we can utilize the Wigner-Eckart theorem to determine the action of $H_I = V_z$.

Note that the dimensionality of the relevant vector space depends on the angular momenta of the multiplets but clearly it is finite. Suppose we consider the two multiplets $|\ell, m\rangle$ and $|\ell+1, m\rangle$; m takes on values from $-\ell$ to $+\ell$ in the first multiplet and from $-\ell-1$ to $\ell+1$ in the second multiplet, so there are $(2\ell+1)+(2\ell+3)=4\ell+4$ of these states.

The action of H_0 on these states is simply

$$L^2|\ell, m\rangle = \ell(\ell+1)|\ell, m\rangle, \tag{23}$$

$$L^2|\ell+1, m\rangle = (\ell+1)(\ell+2)|\ell+1, m\rangle. \tag{24}$$

So all that remains is to determine how V_z acts on these states; here we employ the Wigner-Eckart theorem, which we have extended to the non-Hermitian case as detailed in Appendix. We find $\langle \ell', m'|V_z|\ell, m\rangle = 0$ unless $m = m'$. Thus we need only to determine

$$\begin{aligned}
&\langle \ell, m|V_z|\ell, m\rangle,\\
&\langle \ell+1, m|V_z|\ell+1, m\rangle,\\
&\langle \ell, m|V_z|\ell+1, m\rangle, \text{ and}\\
&\langle \ell+1, m|V_z|\ell, m\rangle
\end{aligned}$$

in order to completely specify V_z in this space. The first two in this list can be expressed in terms of the reduced matrix element α defined in Appendix; in general we find

$$\begin{aligned}
\langle \ell, m|V_z|\ell, m\rangle &= m\alpha_1,\\
\langle \ell+1, m|V_z|\ell+1, m\rangle &= m\alpha_2;
\end{aligned}$$

however we also wish to enforce $\mathcal{P}V_z\mathcal{P} = -V_z$ and $\mathcal{T}V_z\mathcal{T} = -V_z$, which restricts $\alpha_1 = \alpha_2 = 0$. (Determination of $\mathcal{P}$ and $\mathcal{T}$ within this space follows straightforwardly from their action on the spherical harmonics $PY_{\ell m}(\theta,\phi) = (-1)^\ell Y_{\ell m}(\theta,\phi)$ and $TY_{\ell m}(\theta,\phi) = Y^*_{\ell m}(\theta,\phi) = (-1)^m Y_{\ell,-m}(\theta,\phi)$.)

For the other two types of matrix elements, $\langle \ell, m|V_z|\ell+1, m\rangle$ and $\langle \ell+1, m|V_z|\ell, m\rangle$, we find these are proportional to other reduced matrix elements β and γ;

$$\begin{aligned}
\langle \ell+1, m|V_z|\ell, m\rangle &= f_{\ell m}\beta,\\
\langle \ell, m|V_z|\ell+1, m\rangle &= f_{\ell m}\gamma,
\end{aligned}$$

where

$$f_{lm} = \left[\frac{(\ell+1)^2 - m^2}{(2\ell+1)(2\ell+2)}\right]^{1/2}. \tag{25}$$

Note that $f_{\ell m}$ is even in m. When we enforce $PV_zP = -V_z$ and $TV_zT = -V_z$, we find this requires β and γ to be pure imaginary, so we define $\beta = ib$, $\gamma = ic$ for some real numbers

b, c. Note that in determining these matrix elements we do not assume V is Hermitian; we rely only on the commutators of V with the angular momentum operators. (See Appendix for details.)

Now let us write down the matrix corresponding to the Hamiltonian $H = H_0 + H_I$. Consider the two-dimensional subspace spanned by the states $|\ell, m\rangle$ and $|\ell + 1, m\rangle$ for a fixed value of m that lies in the range $-\ell, \dots, \ell$. Within this subspace

$$H_0 = \begin{pmatrix} \ell(\ell+1) & 0 \\ 0 & (\ell+1)(\ell+2) \end{pmatrix} \tag{26}$$

and

$$V_z = \begin{pmatrix} 0 & -icf_{\ell m} \\ -ibf_{\ell m} & 0 \end{pmatrix}. \tag{27}$$

In addition consider the two dimensional subspace spanned by the states $|\ell + 1, \ell + 1\rangle$ and $|\ell + 1, -\ell - 1\rangle$. These states are not coupled by the perturbation V_z to any other state and hence $V_z = 0$ within this subspace. On the other hand the unperturbed Hamiltonian in this subspace is given by

$$H_0 = \begin{pmatrix} (\ell+1)(\ell+2) & 0 \\ 0 & (\ell+1)(\ell+2) \end{pmatrix}. \tag{28}$$

It is convenient to define

$$h_{\ell+1} = \begin{pmatrix} (\ell+1)(\ell+2) & 0 \\ 0 & (\ell+1)(\ell+2) \end{pmatrix} \tag{29}$$

and

$$h_m = \begin{pmatrix} \ell(\ell+1) & -icf_{lm} \\ -ibf_{lm} & (\ell+1)(\ell+2) \end{pmatrix}, \tag{30}$$

where $m = -\ell, \dots, \ell$. The Hamiltonian can now be written as a block-diagonal matrix

$$\begin{pmatrix} h_{\ell+1} & & & & \\ & h_\ell & & & \\ & & h_{\ell-1} & & \\ & & & \ddots & \\ & & & & h_{-\ell} \end{pmatrix}. \tag{31}$$

The individual 2×2 matrices that constitute the Hamiltonian are simple enough that we can obtain analytic expressions for the eigenvalues. The eigenvalues of $h_{\ell+1}$ are two-fold degenerate and are simply $(\ell + 1)(\ell + 2)$. The eigenvalues of h_m are

$$\lambda^{\pm}_{\ell m} = (\ell+1)^2 \pm \sqrt{(\ell+1)^2 - bcf^2_{\ell m}}. \tag{32}$$

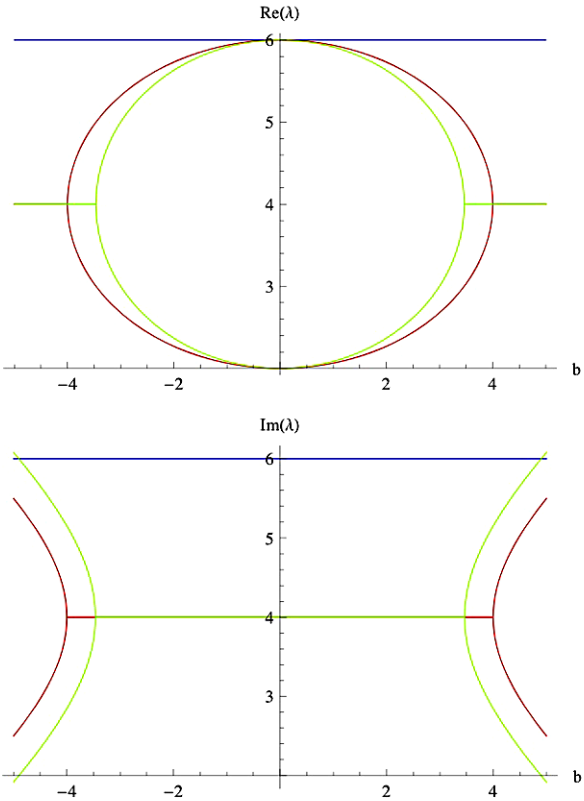

Fig. 2 Real and imaginary parts of the eigenvalues $\lambda_{1,m}$ assuming $b=c$. The *blue line* at $\lambda=6$ corresponds to the 2×2 matrix denoted $h_{\ell+1}$ in the text, with eigenvalues $(\ell+1)(\ell+2)$. The other eigenvalues are m-dependent and correspond to the 2×2 matrices denoted h_m in the text, with eigenvalues given by Eq. (32). As noted $\lambda_{\ell,m}=\lambda_{\ell,-m}$, so there are only two distinct m-dependent curves for $\ell=1$. In each figure, $|m|=1$ is plotted in *red* and $m=0$ is plotted in *green*. Note that the transition to complex eigenvalues occurs at $b=4$ for $\lambda^{\pm}_{1,1}$ and $b=\sqrt{12}\approx 3.47$ for $\lambda^{\pm}_{1,0}$

Note that $\lambda_{\ell,m}=\lambda_{\ell,-m}$ so for all $m\neq 0$ the eigenvalues of h_m are also two-fold degenerate. Clearly, the eigenvalues are real provided

$$bc<\frac{(\ell+1)^2}{f_{lm}^2}. \tag{33}$$

We can make the following observations about the behavior of the eigenvalues. Once $\mathcal{PT}$ symmetry is broken, λ^+ and λ^- form a complex conjugate pair. Since $f_{\ell m}$ has its maximum value for $m=0$, $\lambda^{\pm}$ becomes complex for $m=0$ first. Similarly, $f_{\ell m}$ is minimal for $|m|=\ell$, so $\lambda^{\pm}$ so these are the last eigenvalues to go complex. For example we consider the case $\ell=1$ and choose $b=c$ for simplicity. We plot the eigenvalues in Fig. 2.

It is worth noting that in the Hermitian case $b=-c$. Hence the condition in Eq. (33) that ensures the eigenvalues are real is always met.

4 Conclusion

We conclude by noting two natural generalizations of our results that deserve further investigation. First the model of a particle on an ordinary 2-sphere considered in section II may be generalized to a particle on a sphere in n dimensions. The $\mathcal{PT}$ transition for this model may be amenable to analytic study in the large n limit and may shed some light on $\mathcal{PT}$ symmetric non-linear sigma models of which it would represent a $0+1$ dimensional case [8]. Second the vector model constructed in Sect. 3 may be easily generalized from the symmetry group SO(3) to any Lie group and therefore represents only one member of a large class of such models.

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Appendix: Wigner-Eckart Theorem

Suppose we have an angular momentum operator $\mathbf{L}$ and a vector operator $\mathbf{V}$ satisfying the commutation relations

$$[L_i, V_j] = i\epsilon_{ijk} V_k. \tag{34}$$

Let $|\ell, m\rangle$ denote an angular momentum multiplet of total angular momentum ℓ and z-component m. Then according to the Wigner-Eckart theorem the matrix elements of V_z and $V_\pm = V_x \pm i V_y$ between multiplet states are determined by the commutation relations Eq. (34). In the usual Wigner-Eckart theorem the Cartesian components of the operator $\mathbf{V}$ are assumed to be hermitian. Here we present a non-Hermitian generalization of the theorem.

Following the usual arguments we find the selection rules

$$\langle \ell', m'|V_z|\ell, m\rangle = 0 \quad \text{unless } m' = m, \tag{35}$$

$$\langle \ell', m'|V_+|\ell, m\rangle = 0 \quad \text{unless } m' = m+1, \tag{36}$$

$$\langle \ell', m'|V_-|\ell, m\rangle = 0 \quad \text{unless } m' = m-1. \tag{37}$$

Furthermore the matrix elements vanish unless $\ell' = \ell - 1$ or $\ell' = \ell$ or $\ell' = \ell + 1$.

Consider the case $\ell' = \ell$. Generalization of the usual arguments shows that

$$\begin{aligned} \langle \ell, m+1|V_+|\ell, m\rangle &= A(\ell-m)^{1/2}(\ell+m+1)^{1/2} \quad m = -\ell, \ldots, \ell-1, \\ \langle \ell, m|V_z|\ell, m\rangle &= Am \quad m = -\ell, \ldots, \ell, \\ \langle \ell, m-1|V_-|\ell, m\rangle &= A(\ell-m+1)^{1/2}(\ell+m)^{1/2} \quad m = -\ell+1, \ldots, \ell, \end{aligned} \tag{38}$$

where the proportionality constant A is a complex number called the "reduced matrix element". Note that for $\mathbf{V}$ hermitian, A would have to be real, but there is no such restriction in the non-hermitian case.

Similarly in the case $\ell' = \ell + 1$ we find

$$\begin{aligned} \langle \ell+1, m+1|V_+|\ell, m\rangle &= B\left[\frac{(\ell+m+2)(\ell+m+1)}{(2\ell+2)(2\ell+1)}\right]^{1/2}, \\ \langle \ell+1, m|V_z|\ell, m\rangle &= -B\left[\frac{(\ell-m+1)(\ell+m+1)}{(2\ell+2)(2\ell+1)}\right]^{1/2}, \\ \langle \ell+1, m-1|V_-|\ell, m\rangle &= -B\left[\frac{(\ell-m+1)(\ell-m+2)}{(2\ell+2)(2\ell+1)}\right]^{1/2}, \end{aligned} \tag{39}$$

where $m = -\ell, \ldots, \ell$ and B is another complex reduced matrix element.

Finally in the case that $\ell' = \ell - 1$ we find

$$\begin{aligned}
\langle \ell-1, m+1|V_+|\ell, m\rangle &= -C\left[\frac{(\ell-m-1)(\ell-m)}{(2\ell)(2\ell-1)}\right]^{1/2}, \\
\langle \ell-1, m|V_z|\ell, m\rangle &= -C\left[\frac{(\ell-m)(\ell+m)}{(2\ell)(2\ell-1)}\right]^{1/2}, \\
\langle \ell-1, m-1|V_-|\ell, m\rangle &= C\left[\frac{(\ell+m)(\ell+m-1)}{(2\ell)(2\ell-1)}\right]^{1/2},
\end{aligned} \tag{40}$$

where C is a complex reduced matrix element and $m = -\ell, \ldots, \ell-2$ in the first line of Eq. (40), $m = -\ell+1, \ldots, \ell-1$ in the second line of Eq. (40), and $m = -\ell+2, \ldots, \ell$ in the last line of Eq. (40).

In the hermitian case the reduced matrix elements satisfy $B = C^*$ but in the non-hermitian case there is no such restriction on the complex elements B and C.

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