AIP Journal of Mathematical Physics

Quantum canonical transformations and exact solution of the Schrödinger equation

Ali Mostafazadeh

Citation: Journal of Mathematical Physics **38**, 3489 (1997); doi: 10.1063/1.531864
View online: http://dx.doi.org/10.1063/1.531864
View Table of Contents: http://scitation.aip.org/content/aip/journal/jmp/38/7?ver=pdfcov
Published by the AIP Publishing

Quantum canonical transformations and exact solution of the Schrödinger equation

Ali Mostafazadeh[a)]
Theoretical Physics Institute, University of Alberta, Edmonton, Alberta, T6G 2J1, Canada

(Received 4 December 1996; accepted for publication 25 February 1997)

Time-dependent unitary transformations are used to study the Schrödinger equation for explicitly time-dependent Hamiltonians of the form $H(t)=\mathbf{R}(t)\cdot\mathbf{J}$, where $\mathbf{R}$ is an arbitrary real vector-valued function of time and $\mathbf{J}$ is the angular momentum operator. The solution of the Schrödinger equation for the most general Hamiltonian of this form is shown to be equivalent to the special case $\mathbf{R}=(1,0,\nu(t))$. This corresponds to the problem of a driven two-level atom for the spin half representation of $\mathbf{J}$. It is also shown that by requiring the magnitude of $\mathbf{R}$ to depend on its direction in a particular way, one can solve the Schrödinger equation exactly. In particular, it is shown that for every Hamiltonian of the form $H(t)=\mathbf{R}(t)\cdot\mathbf{J}$ there is another Hamiltonian with the same eigenstates for which the Schrödinger equation is exactly solved. The application of the results to the exact solution of the parallel transport equation and exact holomony calculation for SU(2) principal bundles (Yang–Mills gauge theory) is also pointed out. © *1997 American Institute of Physics.* [S0022-2488(97)01107-9]

I. INTRODUCTION

In nonrelativistic quantum mechanics the dynamics of pure states is determined by the Schrödinger equation,

$$H\psi=i\dot{\psi}, \tag{1}$$

where H is the Hamiltonian, ψ is the state vector representing the state, the dot denotes a time-derivative, and $\hbar$ is set to unity. In general the Hamiltonian may be explicitly time-dependent, in which case the exact solution of the Schrödinger equation is in general not known. In terms of the time-evolution operator $U=U(t)$ defined by $U(t)\psi(0):=\psi(t)$, the Schrödinger equation (1) is written as

$$H(t)U(t)=i\dot{U}(t), \quad U(0)=1. \tag{2}$$

An alternative expression for this equation is $U(t)=\mathscr{T}\exp[-i\int_0^t H(t')dt']$, where $\mathscr{T}$ denotes the time-ordering operator. The purpose of this article is to derive some general sufficiency conditions to obtain the exact solution of Eq. (2) for the dipole Hamiltonians as follows:

$$H(t)=\sum_{a=1}^{3} R^a(t)J_a=\mathbf{R}\cdot\mathbf{J}, \tag{3}$$

where R^a are real functions of time which do not simultaneously vanish, and J_a are generators of the group SU(2) in some irreducible representation.

Following the same line of reasoning as in the Hamilton–Jacobi theory of classical mechanics, one can view the inverse $U^{-1}(t)=U^{\dagger}(t)$ of the evolution operator $U(t)$ as a time-dependent quantum canonical transformation which sets the Hamiltonian to zero. In order to see this more

[a)]Electronic mail: alimos@phys.ualberta.ca

clearly, let us first recall that in quantum mechanics the role of canonical transformations is played by unitary transformations of the Hilbert space. Now consider an arbitrary time-dependent unitary transformation $\psi(t)\rightarrow\psi'(t)=\mathscr{U}(t)\psi(t)$. Requiring this transformation to preserve the form of the Schrödinger equation (1), one has

$$H(t)\rightarrow H'(t)=\mathscr{U}(t)H(t)\mathscr{U}^{\dagger}(t)-i\mathscr{U}(t)\dot{\mathscr{U}}^{\dagger}(t), \tag{4}$$

$$U(t)\rightarrow U'(t)=\mathscr{U}(t)U(t)\mathscr{U}^{\dagger}(0). \tag{5}$$

Hence, $U^{\dagger}(t)$ induces a particular quantum canonical transformation which renders the transformed Hamiltonian zero. In other words, if one views the effect of a quantum canonical transformation as a change of frame in the Hilbert space, then the transformation induced by $U^{\dagger}(t)$ takes one to a moving frame in which the state vector is stationary, i.e., $\psi'(t)=\psi'(0)$.

As it is manifestly seen from Eq. (4), quantum canonical transformations also resemble the non-Abelian gauge transformations of particle physics. Therefore, in a sense solving the Schrödinger equation (2) is equivalent to finding an appropriate gauge in which the state vector is stationary. [Note however that here there does not exist an analog of a non-Abelian gauge symmetry unless one deals with peculiar constraint systems such as those encountered in quantum cosmology.]

In this paper I shall try to demonstrate the utility of this simple observation in solving the Schrödinger equation for a large class of Hamiltonians of the form (3). The basic idea pursued in this paper is to find a series of unitary (gauge) transformations which simplify the form of the Hamiltonian and yield previously unknown exactly solvable cases. Of course *a priori* there is no systematic method of choosing appropriate gauge transformations. However, it turns out that at least for the systems considered here, one is guided by basic group theoretical properties of angular momentum operators and methods of quantum adiabatic approximation. Probably the most notable feature of this method which makes it so effective is its nonperturbative nature.

II. DIPOLE HAMILTONIAN AND PARALLEL TRANSPORTATION IN SU(2) BUNDLES

Consider the Dipole Hamiltonian (3),

$$H=H[R]=\mathbf{R}\cdot\mathbf{J}=r(\sin\theta\cos\varphi J_1+\sin\theta\sin\varphi J_2+\cos\theta J_3)=rW(\theta,\varphi)J_3W^{\dagger}(\theta,\varphi), \tag{6}$$

which describes the dynamics of a magnetic dipole in a changing magnetic field. Here $\mathbf{R}:=(R^1,R^2,R^3)=(r,\theta,\varphi)$ corresponds to the magnetic field vector expressed in units in which the Larmor frequency is set to unity, (r,θ,φ) are spherical coordinates, and

$$W(\theta,\varphi):=e^{-\varphi J_3}e^{-i\theta J_2}e^{i\varphi J_3}. \tag{7}$$

Then an arbitrarily changing magnetic field corresponds to a curve $C:[0,T]\rightarrow\mathbb{R}^3$, $\mathbf{R}=\mathbf{R}(t)=C(t)$.

An application of the dipole Hamiltonian (6) is in the parallel transportation in SU(2) principal fiber bundles (Yang–Mills theory). This is easily seen by recalling that parallel transportation[1] is defined in terms of a Lie algebra-valued one-form (gauge potential) $A=A^a_{\mu}J_a dx^{\mu}$ according to

$$g[\mathscr{C}]=\mathscr{P}\exp\left(-i\int_{\mathscr{C}}A\right)=\mathscr{P}\exp\left(-i\int_{\mathscr{C}(0)}^{\mathscr{C}(\mathscr{T})}A^a_{\mu}J_a dx^{\mu}\right)=\mathscr{T}\exp\left\{-i\int_0^T\dot{x}^{\mu}(t)A^a_{\mu}[x(t)]J_{\alpha}\right\}, \tag{8}$$

where $\mathscr{C}:[0,T]\rightarrow M$ is a curve in the base manifold M of the bundle (space–time in Yang–Mills theory), and t is an arbitrarily chosen parameter of the curve $\mathscr{C}$. It is very easy to recognize the last expression on the right-hand side of Eq. (8) as the time-evolution operator $U(T)$ for a Hamiltonian of the form (6) with $R^a = \dot{x}^\mu(t) A^a_\mu[x(t)]$. Thus an exact solution of the Schrödinger equation for Eq. (6) yields as a special case the solution for the problem of parallel transportation and in particular the calculation of the holonomy elements and Wilson loop integrals in Yang–Mills theory.

Let us next recall the basic properties of the most general Hamiltonians of the form (6).[2] It is not difficult to see that the eigenvalues $E_n[R]$ and eigenvectors $|n;R\rangle$ of $H[R]$ are given by

$$E_n[R] = E_n(r,\theta,\varphi) = E_n(r,0,0) = nr \quad \text{with} \quad n = -j, -j+1, \ldots, j, \tag{9}$$

$$|n;R\rangle = |n;(r,\theta,\varphi)\rangle = |n;(r_0,\theta,\varphi)\rangle = W(\theta,\varphi)|n;(r_0,0,0)\rangle, \quad \theta \in [0,\pi), \quad \varphi \in [0,2\pi), \tag{10}$$

where j corresponds to the spin j-representation of SU(2) and determines the Hilbert space, and $(r_0,\theta_0,\varphi_0) := (r(0)), \theta(0), \varphi(0))$. Hence, the Hamiltonian H is nondegenerate for $r \neq 0$. In order to avoid the complications caused by the sudden collapse of all the energy eigenvalues which occurs at $r=0$, I shall only consider the case where the curve C does not pass through the origin, i.e., $C(t) = \mathbf{R}(t) \in \mathbb{R}^3 - \{0\}$.

Note that $\{|n;(t,\theta,\varphi)\rangle\}$ forms a single-valued orthonormal basis of the Hilbert space for all $\varphi \in [0,2\pi)$ and $\theta \in [0,\pi)$ and that $|n;(r,0,0)\rangle$ are the eigenvectors of $H(r,\theta=0,\varphi=0) = rJ_3$, i.e., $J^3|n;(r,0,0)\rangle = n|n;(r,0,0)\rangle$. For $\theta = \pi, |n;(t,\theta,\varphi)\rangle$ are not single-valued. This is due to the fact that the spectral bundle over $\mathbb{R}^3 - \{0\}$,[3] which yields $|n;(t,\theta,\varphi)\rangle$ as its local basis sections is not trivial. In the parametrization of $\mathbb{R}^3 - \{0\}$ used here the negative z-axis ($\theta = \pi$) is not included in the patch over which $|n;(t,\theta,\varphi)\rangle$ are well-defined. To treat the negative z-axis, one must switch to new coordinates $R' := (r' = r, \theta' = \pi - \theta, \varphi' = \varphi)$. The eigenvectors $|n;R'\rangle$ will then be single-valued everywhere except on the positive z-axis. In the following, I shall assume for simplicity but without loss of generality that the curve C does not intersect the negative z-axis. In the general case where C intersects the negative z-axis, one must make appropriate $U(1)$ gauge transformations which relate $|n;R\rangle$ and $|n;R'\rangle$.[2]

I shall also assume that $|n;(0,0)\rangle$ and therefore $|n;R\rangle$ are eigenvectors of the total angular momentum operator, i.e., the Casimir operator $|\mathbf{J}|^2 = \Sigma_{a=1}^3 J_a^2$. This is always possible unless R^a are also quantized.[2] The latter case will not be considered in the present paper.

III. ADIABATIC APPROXIMATION AND REDUCTION TO TWO-DIMENSIONS

In order to implement the idea of successive quantum canonical transformations, I shall begin using the results of the adiabatic approximation. One knows from the standard arguments of Born and Fock[4] and Kato,[5] that if the time-dependence of the Hamiltonian is adiabatic, then in time the eigenstates of the initial Hamiltonian $H[R(0)]$ evolve into the eigenstates of the Hamiltonian $H[R(t)]$. This is actually very easy to see if one differentiates both sides of the eigenvalue equation

$$H(t)|n;t\rangle = E_n(t)|n,t\rangle, \tag{11}$$

and computes the inner product of both sides of the resulting equation with $|m;t\rangle$ for some $m \neq n$. This yields

$$A_{mn} := \langle m;t|\frac{d}{dt}|n;t\rangle = \frac{\langle m;t|\dot{H}(t)|n;t\rangle}{E_n(t) - E_m(t)}, \quad m \neq n. \tag{12}$$

In Eqs. (11) and (12), $H(t) := H[R(t)]$, $|n;t\rangle := |n;R(t)\rangle$, and $E_n(t) := E_n[R(t)]$. The adiabatic approximation is valid if and only if the right-hand side of Eq. (12) is negligible. Now let us choose $\psi(0) = |n;0\rangle$, then in view of Eq. (12), it is easy to show that $\psi(t) = e^{i\alpha_n(t)}|n;t\rangle$ does solve the Schrödinger equation provided that

$$\alpha_n(t):=\delta_n(t)+\gamma_n(t),\quad \delta_n(t):=-\int_0^t E_n(t')dt',\quad \gamma_n(t):=i\int_0^t A_{nn}(t')dt'. \tag{13}$$

The phase angles $\alpha_n(T)$, $\delta_n(T)$, and $\gamma_n(T)$ for a closed curve C are known as the total, dynamical, and adiabatic geometrical (Berry) phase angles.[6]

The adiabatic approximation which also includes the geometric phase effects corresponds to approximating the time-evolution operator $U(t)$ with

$$U_0(t):=\sum_n e^{i\alpha_n}|n;t\rangle\langle n;0|. \tag{14}$$

In general the approximation $U\approx U_0$ is not valid. However, one can compute U_0 in terms of the eigenvalues and eigenvectors of the Hamiltonian and use $U_0^\dagger$ to perform a quantum canonical transformation. In the remainder of this section, I shall show that indeed this canonical transformation simplifies the form of the Hamiltonian considerably.

In order to do this one must first calculate the matrix elements A_{mn} which enter the calculation of α_n and especially the term $\mathscr{U}\dot{\mathscr{U}}^\dagger$ in Eq. (4) with $\mathscr{U}=U_0^\dagger$. This rather lengthy calculation leads to

$$A_{mn}=i[m(1-\cos\theta)\delta_{mn}+\tfrac{1}{2}\sin\theta(e^{i\varphi}C_m\delta_{mn-1}+e^{-i\varphi}C_n\delta_{m-1n})]\dot{\varphi}$$
$$+\tfrac{1}{2}(e^{i\varphi}C_m\delta_{mn-1}-e^{-i\varphi}C_n\delta_{m-1n})\dot{\theta}, \tag{15}$$

where $C_m:=\sqrt{(j-m)(j+m+1)}=C_{-m-1}$, and extensive use is made of the properties of J_a and $J_\pm:=J_1\pm iJ_2$, particularly

$$e^{-i\beta J_a}J_b e^{i\beta J_a}=\cos\beta J_b+\epsilon_{abc}\sin\beta J_c,\quad a\neq b,$$

$$J_\pm|m;(r,0,0)\rangle=\hbar C_{\pm m}|m\pm 1;(r,0,0)\rangle,$$

where ϵ_{abc} are components of the totally antisymmetric Levi-Civita symbol, with $\epsilon_{123}=1$. Furthermore, one can easily show that $\alpha_n=n\alpha$, $\delta_n=n\delta$, $\gamma_n=n\gamma$, where $\alpha=\delta+\gamma$, and

$$\delta=-\int_0^t r(t')dt',\quad \gamma=-\int_0^t[1-\cos\theta(t')]\dot{\varphi}(t')dt'. \tag{16}$$

These relations are then used to write down the expression for U_0, namely,

$$U_0(t)=W(\theta(t),\varphi(t))e^{i\alpha(t)J_3}W^\dagger(\theta_0,\varphi_0), \tag{17}$$

where W is defined in Eq. (7).

Next let us set $\mathscr{U}=U_0^\dagger$ in Eq. (4). Then using Eq. (15), one finds the expression for the transformed Hamiltonian

$$H_0(t)=\tfrac{1}{2}W(\theta_0,\varphi_0)[\Omega(t)J_++\Omega^*(t)J_-]W^\dagger(\theta_0,\varphi_0), \tag{18}$$

where

$$\Omega(t):=e^{-i[\alpha(t)+\varphi(t)]}[\sin\theta(t)\dot{\varphi}(t)+i\dot{\theta}(t)]. \tag{19}$$

One can easily see that if $\theta_0=\varphi_0=0$, then $W(\theta_0,\varphi_0)=1$ and the expression (18) for the transformed Hamiltonian simplifies considerably. Hence, it is convenient to choose the coordinate system in such a way that $\theta_0=\varphi_0=0$, i.e., $\mathbf{R}(0)=(0,0,r_0)$, or alternatively make a further constant unitary transformation using $\mathscr{U}=W\dagger(\theta_0,\varphi_0)$ which leads to the Hamiltonian

$$H_1(t)=\omega(t)[\cos\,\sigma(t)J_1-\sin\,\sigma(t)J_2], \tag{20}$$

where $\Omega=:\omega e^{i\sigma}$, i.e.,

$$\omega(t):=\sqrt{\dot{\theta}^2+\sin^2\,\theta\dot{\varphi}^2},\quad \sigma(t):=-\alpha-\varphi+\xi \bmod 2\pi,$$

$$\cos\,\xi:=\frac{\sin\,\theta\dot{\varphi}}{\omega},\quad \sin\,\xi:=\frac{\dot{\theta}}{\omega}.$$

One can also combine the two unitary transformations by transforming H by $\mathscr{U}=U_1^\dagger(t)$ with $U_1(t):=U_0(t)W(\theta_0,\varphi_0)$.

The Hamiltonian (20) describes the dynamics of a magnetic dipole in a time-dependent magnetic field which is confined to the $x-y$ plane, i.e., a Hamiltonian of the form (6) corresponding to a planar curve $C_1:[0,T]\rightarrow\mathbb{R}^2-\{0\}$. Hence, the canonical transformation induced by U_1 reduces the three-dimensional problem to a two-dimensional one.

IV. EXACTLY SOLVABLE CASES

Consider the Schrödinger equation for the Hamiltonian H_1. If the angular variable σ happens to be constant, then this equation can be easily integrated. This is simply because in this case H_1 at different times commute and the transformed evolution operator is obtained by its exponentiation, i.e.,

$$U'(t)=e^{-il(t)[\cos\,\sigma_0J_1-\sin\,\sigma_0J_2]}, \tag{21}$$

where $l(t):=\int_0^t\omega(t')dt'$,

$$\sigma_0:=\sigma(0)=-\varphi_0+\xi(0)=-\varphi_0+\tan^{-1}\left[\frac{\dot{\theta}(0)}{\sin\,\theta(0)\dot{\varphi}(0)}\right]=-\varphi_0+\tan^{-1}\left[\frac{\theta'(\varphi_0)}{\sin\,\theta(\varphi_0)}\right], \tag{22}$$

and $\theta':=d\theta/d\varphi$.

Having found the evolution operator U' for H_1, one can use Eq. (5) to write down the solution of the original Schrödinger equation (2). This yields

$$U(t)=U_1(t)U'(t)U_1^\dagger(0)=U_0(t)W(\theta_0,\varphi_0)U'(t)W^\dagger(\theta_0,\varphi_0)\quad\text{for}\quad\sigma(t)=\sigma_0. \tag{23}$$

Note that the parameters σ_0 and l which enter the expression for $U(t)$ are geometric quantities associated with the projection C' of the curve C onto the unit sphere centered at the origin. In particular, l is the length of C'. Furthermore for those portions of the curve C which project to a single point for an extended period of time, ω and consequently H_1 vanish. This is reminiscent of the known fact that the adiabatic approximation is exact when the eigenvectors of the Hamiltonian are stationary.

Another way of arriving at the same conclusion is by performing another quantum canonical transformation with $\mathscr{U}=U_2^\dagger:=e^{-i\sigma(t)J_3}$. This leads to the transformed Hamiltonian

$$H_2=\omega(t)J_1+\dot{\sigma}(t)J_3. \tag{24}$$

Clearly for $\sigma=\text{const}$ the Schrödinger equation for H_2 is exactly solvable. Making a further canonical transformation with $\mathscr{U}=U_3^\dagger:=e^{il(t)J_1}$, one obtains

$$H_3 = \dot{\sigma}[\cos\, l(t) J_3 + \sin\, l(t) J_2] \tag{25}$$

which vanishes identically for $\sigma = \text{const}$. Therefore, as expected the combined transformation $\mathscr{U} = (U_1 U_2 U_3)^\dagger$ leads to a frame in which the Hamiltonian vanishes and the state vector is stationary. Hence, the original time-evolution operator is given by $U = U_1 U_2 U_3$.

Let us next re-express the condition $\sigma = \text{const.}$ in terms of the original variables. Requiring $\dot{\sigma} = 0$, one finds the equivalent condition: $r(t) = r_*(t)$, where

$$r_*(t) := \cos\,\theta\dot{\varphi} - \frac{d}{dt}\left(\frac{\dot{\theta}}{\sin\,\theta\dot{\varphi}}\right) \Big/ \left(1 + \left(\frac{\dot{\theta}}{\sin\,\theta\dot{\varphi}}\right)^2\right) = \left[\cos\,\theta - \frac{d}{d\varphi}\left(\frac{\theta'}{\sin\,\theta}\right) \Big/ \left(1 + \left(\frac{\theta'}{\sin\,\theta}\right)^2\right)\right]\dot{\varphi}. \tag{26}$$

Therefore, one has:

Lemma 1: The exact solution of the Schrödinger equation (2) is given by Eq. (23) provided that the magnitude of the magnetic field depends on its direction according to $r(t) = r_*(t)$.

This is quite remarkable, for it indicates that for every Hamiltonian of the form (6) for which r_* does not vanish for extended periods of time, there exists another Hamiltonian with the same eigenvectors [note that the eigenvectors only depend on the direction of the magnetic field] whose Schrödinger equation is exactly solvable. Note that for time intervals during which $r_* < 0$, one can consider the time-reversed system where $r_* > 0$. The evolution operator obtained for the time-reversed system yields the original time-evolution operator upon inversion. This leaves only the cases where r_* vanishes, i.e., either $\dot{\varphi} = 0$ or $\theta' = \sin\theta \tan[\sin\,\theta + c]$ for some constant c. A simple case where the latter equation is satisfied is $\theta = \pi/2$ and $c = -1$. This means that for the planar curves with $\theta = \pi/2$ such as C_1, one cannot enforce the condition $r = r_*$ and the exact solution cannot be obtained in this way. Therefore a direct repetition of the same procedure for the Hamiltonian H_1 will not lead to the exact solution. In the remainder of this section I shall demonstrate, however, that by a straightforward redefinition of the time one can generalize Lemma 1 further.

Let us first note that for the case where $\omega = 0$ the exact solution is given by the adiabatic approximation. Hence, without loss of generality one can restrict to the case $\omega \neq 0$. In this case the length l of the projection C' of the curve C is a monotonically increasing function of time t. Therefore it can be used to parametrize the evolution of the system, i.e., replace t. Changing variables from t to l in the Schrödinger equation for the Hamiltonian H_1 and making use of $\omega \neq 0$, one has

$$\overline{H_1}(l)\overline{U_1}(l) = i\,\frac{d}{dl}\,\overline{U_1}(l), \tag{27}$$

where

$$\overline{H_1}(l) := \cos\,\sigma(l) J_1 - \sin\,\sigma(l) J_2 = e^{i\sigma(l)J_3} J_1 e^{-i\sigma(l)J_3}. \tag{28}$$

This reduces the problem to the case of a magnetic field which traces a circular path in the $x-y$ plane with an angular frequency, $\nu := d\sigma/dl = (r - r_*)/\omega$. Note that the presence of $\omega(t)$ on the right-hand side of Eq. (20) is quite essential in the redefinition of time.

Let us next transform to the rotating frame defined by $\mathscr{U} = \overline{U_2^\dagger}(l) := e^{-i\sigma(l)J_3}$. In view of Eq. (4), this leads to the transformed Hamiltonian

$$\overline{H_2}(l) = J_1 + \nu(l) J_3, \tag{29}$$

which describes a magnetic field with a constant x-component and a variable z-component. Such systems are widely encountered in the study of nuclear and optical magnetic resonance. For a recent study of an iterative solution of the Schrödinger equation for this Hamiltonian see Refs. 7 and 8.

Note that for $\nu=\nu_0=\text{const.}$, $\overline{H_2}$ is constant. Hence, the transformed time-evolution operator is given by $\overline{U_3}(l):=\exp[-il(J_1+\nu_0 J_3)]$, and one has

$$\begin{aligned} U(t) &= U_1(t)\overline{U_2}(l(t))\overline{U_3}(l(t))\overline{U_2^\dagger}(0)U_1^\dagger(0) \\ &= U_0(t)W(\theta_0,\varphi_0)\overline{U_2}(l(t))\overline{U_3}(l(t))\overline{U_2^\dagger}(0)W^\dagger(\theta_0,\varphi_0). \end{aligned} \tag{30}$$

This concludes the derivation of the exact solution of the Schrödinger equation for the case where $\sigma(l)=\sigma_0+\nu_0 l$, alternatively, $r(t)=r_*(t)+\nu_0\omega(t)$. This is a generalization of Lemma 1. It states that even for the time periods during which $r_*=0$, the above procedure still leads to exactly solvable Schrödinger equations. More precisely, the following lemma holds.

Lemma 2: The exact solution of the Schrödinger equation (2) is given by Eq. (30), provided that the magnitude of the magnetic field depends on its direction according to $r(t)=r_*(t)+\nu_0\omega(t)$, for some constant ν_0.

A direct consequence of this result is

Corollary: For every Hamiltonian of the form (6), there exists another Hamiltonian with the same eigenvectors for which the Schrödinger equation is exactly solvable.

V. CONCLUSION

In this paper, I have used a variety of time-dependent unitary transformations of the Hilbert space to obtain the exact solution of the Schrödinger equation for a large class of explicitly time-dependent dipole Hamiltonians. This involved redefinition of the time variable which was a consequence of transforming to a moving frame via the inverse of the adiabatically approximate time-evolution operator. In this frame the natural choice for the evolution parameter turned out to be the length of the projection of the curve C traced by the tip of the magnetic field onto the unit sphere centered at the origin.

The reduction of the general problem to that of the Hamiltonian $\overline{H_2}=J_1+\nu(l)J_3$ may also be used to set up an approximation scheme for large ω. This is due to the fact that $\nu=(r-r_*)/\omega$ may be neglected for large ω, in which case Lemma 2 provides the solution.

This is particularly effective for the dipole Hamiltonians which correspond to a planar curve C, for which $r_*=0$, e.g., $\overline{H_1}$. For these Hamiltonians, the approximation is valid if the parameter $r(t)/\omega(t)$ is negligible. Note also that for such Hamiltonians if $r(t)$ and $\omega(t)$ are proportional, then Lemma 2 yields the exact solution to the Schrödinger equation.

Moreover, by successive application of the method used in this reduction, i.e., by replacing the original Hamiltonian H by $\overline{H_1}$ and repeating the same analysis, one obtains an iterative solution of the Schrödinger equation which yields a product expansion of the time-evolution operator. The condition of the termination of this expansion after a finite number of iterations may seem to lead to (possibly) more general exactly solvable cases. It turns out that this is in fact not the case. This is because enforcing the condition that the above expansion be terminated after the second iteration leads to $\omega=\text{const.}$, which is certainly not more general than the conditions of Lemma 1 and Lemma 2. This marks a unique property of the Hamiltonians of type $H(t)=J_1+R^3(t)J_3$.

The results of this paper have direct applications in the computation of the holonomy elements and Wilson loop integrals in Yang–Mills theory where the gauge group is SU(2). [Clearly the U(2) case can also be handled similarly.] In this case the original parameters R^a of the Hamiltonian (6) are identified with $\dot{x}^\mu A_\mu^a$, where (A_μ^a) corresponds to the local connection one-form (gauge potential) and the gauge transformations correspond to quantum canonical transformations of the associated Hamiltonian. Another area of application of the results of this paper is in the calculation of non-Abelian U(2) geometric phases[9] such as those encountered in the study of the three-level systems.[10]

Note added. In Ref. 11, Berry has introduced an iterative procedure to compute corrections to the adiabatic geometric phase. Although this procedure also makes use of quantum canonical transformations, unlike the method described in this paper it is perturbative in nature. In particular, it cannot be used to yield exact solutions of the Schrödinger equation.

ACKNOWLEDGMENTS

I would like to thank Dr. M. Razavi for invaluable comments and suggestions and acknowledge the financial support of the Killam Foundation of Canada.

[1] M. Nakahara, *Geometry, Topology and Physics* (Hilger, London, 1990).
[2] A. Bohm, *Quantum Mechanics: Foundations and Applications*, 3rd ed. (Springer, New York, 1993), Chaps. 22 and 23; see also Ref. 12.
[3] B. Simon, Phys. Rev. Lett. **51**, 2167 (1983); see also Ref. 12.
[4] M. Born and V. Fock, Z. Phys. **51**, 165 (1928).
[5] T. Kato, J. Phys. Soc. Jpn. **5**, 435 (1950).
[6] M. V. Berry, Proc. Soc. London, Ser. A **392**, 45 (1984).
[7] A. Muriel, Phys. Rev. A **50**, 4286 (1994).
[8] A. Royer, Phys. Rev. A **54**, 3685 (1996).
[9] F. Wilczek and A. Zee, Phys. Rev. Lett. **52**, 2111 (1984).
[10] A. Mostafazadeh, University of Alberta Preprint No. THY 29-96, quant-ph/9608031.
[11] M. V. Berry, Proc. R. Soc. London, Ser. A **414**, 31 (1987).
[12] A. Bohm, A. Mostafazadeh, and J. Zwanziger, *Quantum Geometric Phase: Theory and Experiment* (Springer, Berlin, in press).