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THE ANALYTIC STRUCTURE OF MANY-BODY PERTURBATION THEORY

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Abstract: The Goldstone linked cluster expansion is used to determine the energy as an analytic function of the coupling constant. This function is many-valued and describes the various energy levels of the system. The energy of each level can be obtained from the Goldstone expansion by continuing it analytically along a properly chosen path in the complex plane. The Brueckner ladder approximation [3]) is shown to be an approximation to an analytic continuation along a path which always leads to the normal state — the state in which no binding occurs.

1. Introduction

It is well-known that convergence difficulties arise in the Goldstone [1]) expansion (also known as linked cluster expansion) for the ground state energy of a many-fermion system. Sometimes it is even claimed that this series never converges [2]). Such claims generally follow the treatment of an infinite system with a continuous energy spectrum. In this paper finite systems with dicrete energy spectra are considered. In sect. 6 it is shown that the radius of convergence of the Goldstone series as a power series in the coupling constant g tends to zero as the volume of the system becomes infinite. For any finite volume, however, the radius of convergence is finite but small. For sufficiently large volumes it is so small that for the actual value of g the series fails to converge. In most macroscopic systems this is the case.

In most applications in physics a power series is used as a basis for a class of approximations. The approximations are the sums of the first few terms of the series. In order for these approximations to be valid it is necessary that the series converge. The Goldstone expansion in a large system is useless in this respect. A power series, however, means more than a formula from which numerical values can be obtained. A power series defines an analytic function which may be meaningful also far away from the region of convergence of the series. It is argued in sect. 2 that the analytic function of g defined by the Goldstone series is a multivalued one, and its different values on the real axis

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correspond to all the energy levels of the system †. The Goldstone series therefore defines all the energy levels of the system for all values of g. In order to obtain them it is necessary to perform analytic continuation from the region of convergence of the series to the desired value of g along a properly chosen path in the complex plane.

The exact analytic continuation is a process which is as complicated as the summation of the expansion to all orders. Approximation methods for the analytic continuation along certain paths, however, may be of practical value. It is shown that the Brueckner [3]) ladder approximation can be regarded as such an approximation. The analytic continuation which it approximates is along a path which always leads to the "normal state" of the system. This is shown in model calculation in sect. 5. In sect. 6 the same is established for a general many-fermion system treated by the Bethe-Goldstone [4]) method. This method, although inconsistent for treatment of the ground state [5]), is justified for the normal state.

The difference between our description of the ladder approximation and the usual one is that we regard it as an approximation to an analytic continuation whereas usually it is presented as an analytic continuation of an approximation. Usually it is argued that the ladder diagrams are the largest contributions to the perturbation series in every order. This establishes the ladder approximation within the region of convergence of perturbation theory. The ladders are, however, used outside this region of convergence where some partial sums of ladders (the ones with holes close to the Fermi surface) diverge. What one uses is therefore the analytic continuation of the ladder approximation. The analytic continuation of an approximation makes very little sense because an approximation which reproduces the numerical values of a function in a certain region does not necessarily reproduce its analytic properties. Thus, the ladders which approximate the ground state of a system inside the region of convergence are a single-valued function while the true energy function is many-valued. In our way of justification, the ladders are regarded as the first term in a convergent series (which is roughly an expansion in the density of the system). The region of convergence of this expansion dictates the path of analytic continuation in the complex plane and determines which level is approximated by the ladders outside the region of convergence.

Although it should be possible in principle to obtain the energy of any level of the system starting from the Goldstone expansion, we are unable to offer a workable approximation to any such continuation different from the one represented by the ladders. In particular it would be interesting to find the way to approximate the ground state. The ground state in the case of attractive interaction may be of the "superconducting" type. The formulation of an

† All those which belong to the same values of conserved quantities (e.g. total momentum) as the ground state.

approximation to the ground state starting from the Goldstone expansion is therefore also a derivation of superconductivity in terms of Feyman diagrams. We cannot at present formulate such an approximation. The path of analytic continuation at which one should aim in order to obtain the ground state is, however, evident (see sect. 3); it is the real axis. Any summation of the series of diagrams which is to lead to the ground state must therefore be free of singularities on the negative part of the real g axis. The ladders violate this requirement.

2. The Analytic Structure of the Energy Function

The energy of a quantal system is obtained by the diagonalization of a matrix. This matrix is in general of infinite order. We shall content ourselves here to examine the properties of energy levels derived from a mtrix of order n, hoping that the general features will persist even as $n \to \infty$. (Another possible point of view is the assumption that the real problem could be approximated to any desired accuracy by replacing the infinite matrix by a large finite matrix. Our arguments would then be valid for all such approximations). We shall limit ourselves to systems of finite size and therefore the spectrum will be discrete and no continuous matrices will be considered.

The Hamiltonian matrix generally depends on a number of parameters, such as, for example, the coupling constant g of the interaction between particles and the density ρ of the particles. The matrix elements are generally simple analytic functions of these parameters (e.g. they are usually linear functions of g), so that the same (analyticity, not linearity) would hold for the eigenvalues. The eigenvalues obtained by solution of an algebraic equation are therefore also analytic functions of the parameters except at a set of isolated singular points. Let us now consider the secular equation

$$|H(g)-E| \equiv P(E) = 0. \tag{2.1}$$

This is an algebraic equation of the nth order, P being a polynomial of that order. It therefore has in general n distinct solutions $E_i(g)$, $i = 1, \ldots n$. The polynomial $P(E)$ can be expressed in terms of these functions as

$$P(E) = \prod_{i=1}^{n} (E-E_i(g)). \tag{2.2}$$

The functions $E_i(g)$ are analytic functions of g. From the hermiticity of H it follows that the E_i are real for real values of g. The E_i represent the energy levels of the system. We shall further assume that these levels do not cross †,

† It is of course possible to construct examples in which levels do cross. Such a degeneracy can always be removed by an infinitesimal change in the interaction. Crossings do occur of course for levels belonging to different values of conserved quantities (total momentum, total angular momentum etc.). In this case the conservation law which we do not wish to violate forbids the small change in the interaction.

which is the general case, i.e.

$$E_1(g) < E_2(g) < E_3(g) \ldots < E_n(g) \quad \text{for all real values of } g.$$

We therefore assume $E_i(g) \neq E_j(g)$ for $i \neq j$ and for real g. On the other hand the equation $E_i(g) = E_j(g)$ for given values of i and j must have a solution g_{ij} in the complex plane. This follows from the analyticity of the function $E_i(g) - -E_j(g)$ which must therefore assume every value including zero at least once (excluding the trivial case $E_i(g) - E_j(g) = \text{constant}$). We can further see that $E_i(g_{ij}) = E_j(g_{ij})$ implies also $E_i(g_{ij}^*) = E_j(g_{ij}^*)$, where the star denotes the complex conjugate. (This is so because $E(g)$ is real for real g and therefore $E(g^*) = E^*(g)$.)

Let us now examine the behaviour of the energy functions in the neighbourhood of the points g_{ij} in the complex g plane for which

$$E_i(g_{ij}) = E_j(g_{ij}) \equiv E_{ij}. \tag{2.3}$$

From the form (1.2) for $P(E)$ it follows that

$$P(E_{ij}) = \frac{\mathrm{d}P(E)}{\mathrm{d}E}\bigg|_{E=E_{ij}} = 0. \tag{2.4}$$

Therefore, if we expand $P(E)$ around the values $g = g_{ij}$, $E = E_{ij}$, the expansion would be

$$P(E) = \tfrac{1}{2}(E-E_{ij})^2 \frac{\mathrm{d}^2 P(E_{ij})}{\mathrm{d}E_{ij}^2} + \ldots + (g-g_{ij}) \frac{\partial P(E_{ij})}{\partial g}\bigg|_{g=g_{ij}} + \ldots . \tag{2.5}$$

Confining ourselves to a small neighbourhood of the values g_{ij} and E_{ij} we may omit the higher order terms. Imposing eq. (2.1) we find an approximate solution for the neighbourhood of g_{ij} which is

$$E = E_{ij} \pm \sqrt{g-g_{ij}} \, \frac{2\partial P/\partial g}{\mathrm{d}^2 P/\mathrm{d}E^2}\bigg|_{\substack{g=g_{ij} \\ E=E_{ij}}}. \tag{2.6}$$

The two solutions in eq. (2.6) correspond to E_i and E_j, both of which become E_{ij} when $g = g_{ij}$. We immediately see that g_{ij} is a singularity of both solutions; it is the branch point in which the i and j sheets of the Riemann surface corresponding to the solution of eq. (2.1) meet. Starting with E_i and going round the point g_{ij} we would end with E_j and vice versa. In each sheet corresponding to any level E_i there are $n-1$ such branch points corresponding to all other levels E_j to which one can pass by going around the branch point. Together with each branch point g_{ij} in the upper half plane there exists a branch point at g_{ij}^* in the lower half plane. This situation is described in fig. 1. Going around both branch points one returns to the same value.

It is of course possible that three levels would coincide at some point in the complex plane which would be a double branch point. We shall not consider this possibility which may be viewed as the coincidence of two simple branch points.

Let us now consider the situation, shown in the left hand part of fig. 1, when g_{ij} it close to the real axis and therefore g_{ij} and g_{ji}^* are close together. An approximate form of E_i and E_j in this neighbourhood, taking both branch points into account, is

$$E = \mathrm{Re}\, E_{ij} + \frac{g - \mathrm{Re}\, g_{ij}}{\mathrm{Im}\, g_{ij}} \mathrm{Im}\, E_{ij} \pm A\sqrt{(g-g_{ij})(g-g_{ij}^*)}. \qquad (2.7)$$

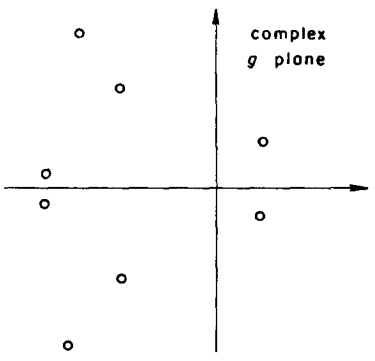

Fig. 1. Branch points between the different energy levels occur in pairs at conjugate points of complex g plane, off the real axis.

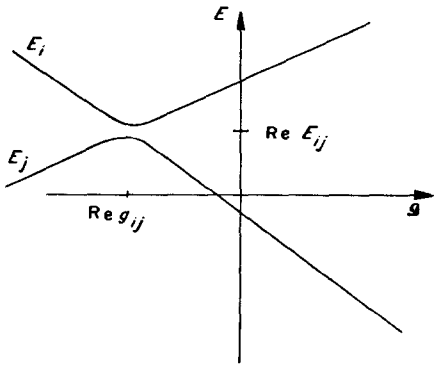

Fig. 2. An "almost crossing" of the i and j levels for real g.

Let us now plot the values of E_i and E_j for real g with the aid of (2.7). The result is shown in fig. 2. In the neighborhood of $\mathrm{Re}\, g_{ij}$, E_i and E_j form the two branches of a hyperbola. They are two levels which "almost cross" but at the last moment change names instead. We see that the well known phenomenon of levels "almost crossing" corresponds to a pair of branch points which are very close to the real axis. The distance between the branch points can serve as a measure of the sharpness of the "almost crossing". In the limiting case in which the two branch points coincide on the real axis the levels actually cross.

In the next section we deal with the physical aspects of this "almost crossing" of levels.

3. Normal State, Cooper Pairs and Ground State

It has been shown by Cooper [5]) that in the presence of an interaction and of a filled Fermi sphere, there exist pair states which coherently make use of the interaction and the free phase space outside the Fermi sphere in order to have an energy and momentum distributions very different from the unperturbed

ones. A many-fermion system may have any number of pairs in Cooper type states. We shall refer to the state of the system with no Cooper pairs as the *normal state*. The ground state on the other hand may have a number of Cooper pairs depending on the strength of the interaction. For strong attractive interactions the ground state is of the "superconducting" type which means that all particles form Cooper pairs [7]). Fig. 3(a) describes schematically the energies of levels having different numbers of Cooper pairs as a function of the coupling constant g. In fig. 3(b) we make the refinement of noting that the different levels do not actually cross.

The normal state is the ground state around $g = 0$ and there it is calculable by means of the Goldstone perturbation method. A little more will be said about

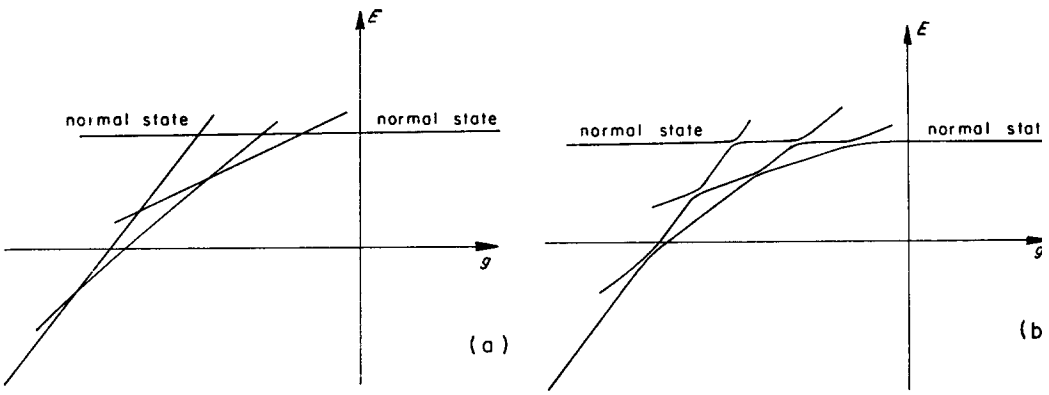

Fig. 3. (a) As the interaction becomes attractive the normal state is crossed by a state in which a pair is bound which becomes the ground state. This state is in turn crossed by a state with two bound pairs and so on. (b) The crossings of fig. 3 (a) are replaced by almost crossings.

this method in the next section. Here it is enough to note that the perturbation series ceases to converge before the first almost crossing to the left of $g = 0$. This is because a power series in g describing the energy function can converge only as far as the nearest singularity. The "almost crossing" is accompanied as we have noticed in the previous section by two branch points on both sides of the real axis near the point of the almost crossing. The power series cannot converge beyond these singularities. Therefore any attempt to calculate any energy level of the system to the left of the first crossing based on perturbation theory is an attempt at analytic continuation. The actual level which such a method would describe depends on the path along which the continuation is performed.

We now define the paths along which the continuation should be performed in order to end up with the ground state or with the normal state. In order to obtain the ground state energy one should perform the analytic continuation along the real axis. Starting with the lowest level for $g = 0$ one would always

be describing the lowest level because for real g levels never cross. In order to obtain the normal state one must go around the branch point corresponding to each "almost crossing". A possible prescription for doing this would be to go up along the imaginary axis a distance corresponding to the distance between the branch points and the real axis, then to continue parallel to the real axis and finally to return to it at the desired point. The distance from the real axis which we choose defines the sharpness of the "almost crossings" at which we want to cross and in fact defines what we mean by the normal state. The paths are shown in fig. 4.

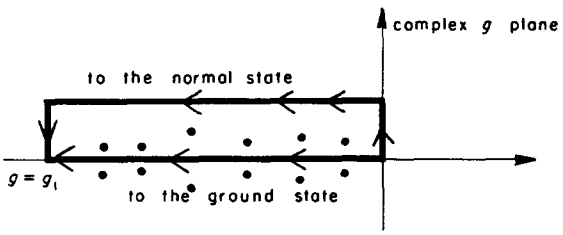

Fig. 4. The paths for analytic continuation which lead to the energy of the ground state and the normal state for $g = g_1$.

4. The Goldstone Perturbation Series

The Goldstone perturbation theorem [1]) states that the energy of the lowest state of the system which is not orthogonal to the unperturbed ground state — which in general is the true ground state — is given by the expression

$$E-U = \left\langle v \sum_n \left(\frac{1}{U-H_0} v\right)^n \right\rangle_c , \tag{4.1}$$

as long as this series converges. Here E, is the true energy, U the unperturbed energy, H_0 the unperturbed Hamiltonian and v the interaction. The notation $\langle\ \rangle_c$ means that only contributions of connected vacuum-vacuum diagrams should be taken into account.

It should be noted that the right hand side of eq. (4.1) is actually a double series since in every order of perturbation theory there appear a number of diagrams. The name "perturbation theory" suggests the following order of summation: first sum together the contributions in each order in the coupling constant g, thus obtaining a power series in g; then sum this power series. In this order of summation the first summation is a trivial one since it is a finite summation in every order and no convergence problems arise. The second summation, however, is very complicated and there is no practical possibility of carrying it out to all orders. All we can do is approximate the infinite sum by the sum of the first few terms. This approximation is valid only within the radius of convergence of the power series in g which is, following the consider-

ations of the previous section, very small. There is no hope of obtaining any results for large values of g by this method.

Let us now consider another order of summation of the double series which will prove more fruitful. Following Brueckner [3]) let us arrange the diagrams according to the number of interactions involving holes. Let us first sum together all diagrams with a given number of such interactions. The momentum of holes must be integrated over the Fermi sphere, the volume of which is proportional to the density of particles in the system. The new series obtained by summing together all diagrams with the same number of hole interactions is therefore roughly an expansion of the energy of the system in powers of its

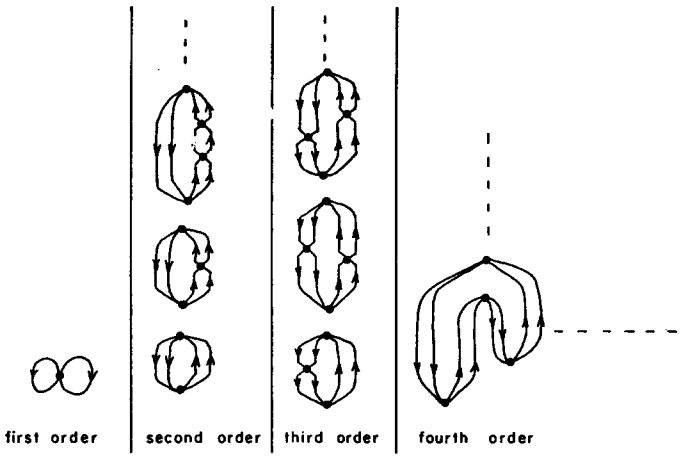

Fig. 5. The grouping together of diagrams according to the orders of the terms of the second summation. Because of the great variety already in fourth order only one diagram of this order out of the many types possible is shown.

density. This grouping of diagrams is indicated in fig. 5. It is seen that in the first stage we are summing over repeated scatterings along the particle lines, i.e. summing ladders. In this order of summation both the first summation and the second one are infinite sums. But the summation of ladders is an infinite sum of a particularly simple form. In essence it is a geometric progression or rather a sum of a great number of geometric progressions in the general case. The summation of ladders can be performed to all orders and the result can be obtained in closed form. This closed form has no other singularities than a number of poles on the real axis and it is valid for all values of g. The second summation, that of the power series in the density, is of a far more complicated nature and we cannot hope to perform it to all orders. All we can do for the second summation is to approximate the infinite sum by the sum of the first few terms. This again is valid only within the region of convergence of the second summation. But the region of convergence of the second summation in

this order of summation differs from that of the second summation in the previous perturbation theory. Having "absorbed" some of the convergence difficulties into the first sum — the sum of ladders — we may hope that the region of convergence of the second summation might become larger than in simple perturbation theory. We shall check this hope for a simple model in the next section and later return to the general case.

5. Model Calculation

Let us consider a system consisting of two dingle particle states and populated by one particle. Let the unperturbed hamiltonian be

$$H_0 = \begin{bmatrix} \varepsilon & 0 \\ 0 & 0 \end{bmatrix}, \quad \varepsilon > 0, \tag{5.1}$$

and let the perturbation be a single particle force of the form

$$v = \begin{bmatrix} 0 & v \\ v & 0 \end{bmatrix}. \tag{5.2}$$

This model has been considered in detail elsewhere [8]) and the summation of diagrams has been explicitly carried out to all orders. The result for the ground state energy has been as expected

$$E = \tfrac{1}{2}\varepsilon \left(1 - \sqrt{1+4\frac{v^2}{\varepsilon^2}}\right). \tag{5.3}$$

Let us now turn the two states into two groups of degenerate states, the upper one being K-fold degenerate and the lower one M-fold degenerate. We consider the system populated by M fermions. (The identity of the number of particles with the degeneracy of the lower state is essential since one of the necessary conditions for the Goldstone theorem to hold is the non-degeneracy of the unperturbed ground state of the *system*). As for the perturbation we shall choose it again of the form (5.2), i.e. it can take the particle from any of the lower states to any of the upper states with a matrix element v. The energy in our new many-particle system is

$$E = \tfrac{1}{2}\varepsilon \left(1 - \sqrt{1+4KM\frac{v^2}{\varepsilon^2}}\right). \tag{5.4}$$

The possible diagrams in our system display only creation and annihilation of pairs at the interaction points. There is no ascattering. These diagrams are shown in fig. 6. We shall now introduce particle scattering by considering a

perturbation of the form

$$v = \begin{pmatrix} v \ldots v & v \ldots v \\ \vdots \quad \vdots & \vdots \quad \vdots \\ v \ldots v & v \ldots v \\ v \ldots v & 0 \ldots 0 \\ \vdots \quad \vdots & \vdots \quad \vdots \\ v \ldots v & 0 \ldots 0 \end{pmatrix} \begin{matrix} \} K \\ \\ \\ \} M \end{matrix} \qquad (5.5)$$

$$\underbrace{\quad}_{K} \quad \underbrace{\quad}_{M}$$

Here the perturbation can take a particle not only from the lower states to the upper states and back, but also from any of the upper states into any

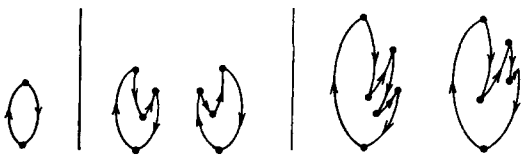

Fig. 6. Grouping of diagrams to form terms of the perturbation expansion in the model (no scattering).

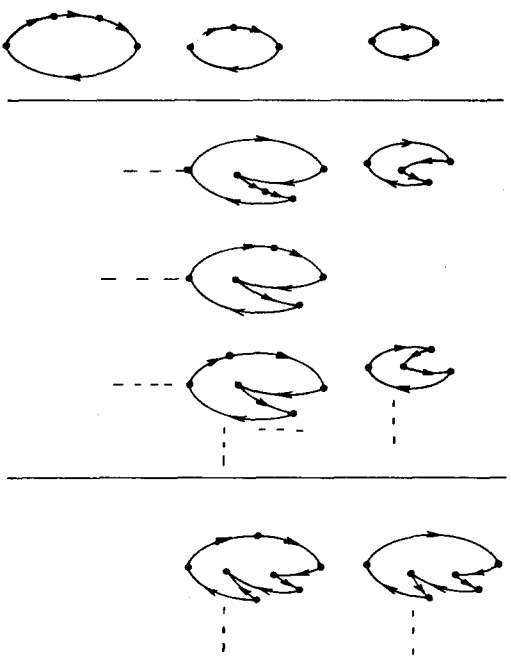

Fig. 7. Grouping of diagrams to form terms of the second summation in the model (scattering included).

other (including itself). This change gives rise to many new diagrams. The diagrams which should now be considered are indicated in fig. 7. The diagrams in fig. 7 are grouped together according to the number of hole interactions in the Brueckner fashion. They differ from the diagrams in fig. 6 by the addition of any number of scatterings along the particle lines. Let us sum over these scatterings (ladders) first.

The propagator for a particle line, which had been $1/(0-\varepsilon) = -1/\varepsilon$, now becomes (including any number of scatterings)

$$-\frac{1}{\varepsilon} + \left(-\frac{1}{\varepsilon}\right) Kv \left(-\frac{1}{\varepsilon}\right) + \ldots = -\frac{1}{\varepsilon+Kv}. \tag{5.6}$$

The new expression for the energy can therefore be obtained from (4.4) by the substitution

$$\varepsilon \to \varepsilon+Kv, \tag{5.7}$$

which yields

$$E = \tfrac{1}{2}(\varepsilon+Kv)\left(1-\sqrt{1+\frac{4KMv^2}{(\varepsilon+Kv)^2}}\right). \tag{5.8}$$

By this substitution we have implicitly performed the "second summation", having performed the sum of ladders explicitly in (5.6). Let us now consider the convergence problems of both summations. The left hand side of eq. (5.6) is a geometric progression whose sum has a pole at $v = -\varepsilon/K$. It is seen that for large K the region of convergence of (5.6) is very limited. On the other hand the closed expression (5.6) is the analytic continuation of the sum for all values of v and is valid everywhere. Let us now consider the second summation. An expansion of (5.8) in terms of interactions involving holes is its expansion in powers of those v that do not appear in the expression $\varepsilon+Kv$. Equivalently, it is the expansion of (5.8) in terms of $4KMv^2/(\varepsilon+Kv)^2$. The region of convergence of this expansion may be readily determined. However, we shall first deduce the form of this region by an approximate argument which applies also in the general case.

We shall first note that at $v = -\varepsilon/K$ the second summation certainly fails to converge since each term contains the "new particle propagator" (5.6) to the power of the number of hole lines, and the new propagator is infinite there. We then notice that each term of the second summation depends on v through the "new propagator" and also directly. As the new propagator is singular at $v = -\varepsilon/K$ we shall assume that in the neighbourhood of this point in the v plane the dependence on v through the new propagator is the most important one. We shall neglect the direct dependence of the terms of the series on v. This turns the series into a power series in the new propagator which converges when $\varepsilon+Kv$ is large enough. The boundary of the region of convergence is decided by the singularities of the function (5.8).

The function (5.8) is a two-valued function and has branch points where the square root vanishes. This occurs at

$$v = -\frac{\varepsilon}{K \pm 2i\sqrt{KM}}. \tag{5.9}$$

Let us consider the case $K \gg M$ which corresponds to the case of low density (since M represents the volume of the Fermi sphere). In this case the branch points lie very close to the point $v = -\varepsilon/K$ at which the new propagator has its pole. It is therefore reasonable to assume that up to the branch points the dependence of the terms of the second summation on v through the propagator is the dominant one, and that the series is an expansion in powers of the propagator. It would follow from this that the second summation would converge everywhere outside a circle the centre of which is at $v = -\varepsilon/K$ and which passes through the branch points (5.9). If the direct dependence on v is taken into account as well, one still expects the second summation to converge outside a region which contains the pole of the propagator and on the boundary of which the branch points lie.

We can now go back to (5.8) and check what the region of convergence of the expansion of the root in powers of $4KM\,v^2/(\varepsilon+Kv)^2$ is. It turns out that this expansion converges everywhere outside a circle which contains the pole $v = -\varepsilon/K$ (although not at the centre) and which passes through the branch points. The region of convergence is shown in fig. 8. The pole is at P, the

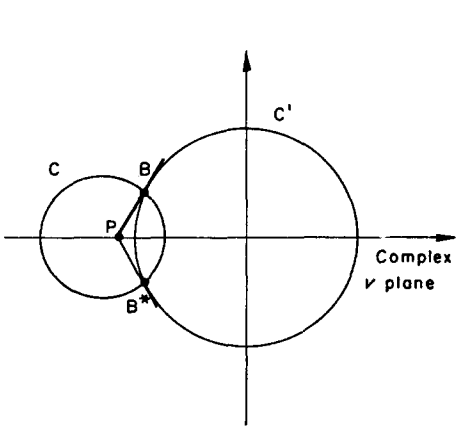

Fig. 8. The analytic properties of the model in the complex v plane. P is the pole of the new propagator. The branch points are at B and B*. The perturbation expansion converges inside C'. The second summation converges outside C.

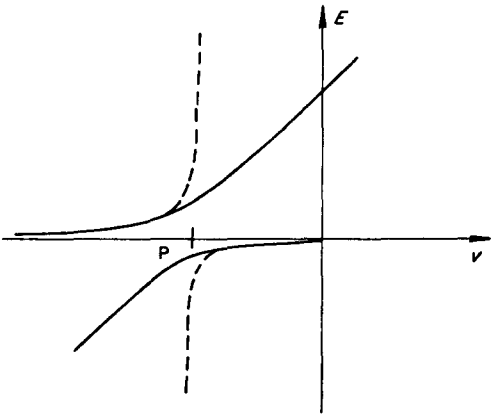

Fig. 9. The curves of the energy levels of the system versus v are shown. The dotted lines correspond to the ladders which approximate the normal state except near P.

branch points at B and B* and the second summation converges outside the circle C.

Let us now return to the usual perturbation expansion in which diagrams are grouped together according to the number of interactions. This is just the expansion of (5.8) in powers of v. It converges for small v up to the first singularities which are just the branch points at B, B*. The region of convergence of the perturbation expansion is therefore the inside of the circle C'. It can be checked that the tangents to the circle C' through P touch it at B and B*.In the case of low density, $M \ll K$, the pole P lies very close to the circle C', the branch points are close together and the circle C is very small. Perturbation theory converges only within C'. The second summation in the Brueckner order, however, converges almost everywhere. It fails to converge only in a very small region around P. Wherever the second summation converges we may replace it by the sum of the first few terms or even by the first term alone which is the sum of ladders. This term is

$$KMv^2/(\varepsilon+Kv), \tag{5.10}$$

and it approximates (5.8) well provided we keep away from the pole where $\varepsilon+Kv = 0$.

We have thus obtained a good approximation for the energy of a level of the system. It remains to be established which level it is. The two branch points at B, B* which lie close together indicate an "almost crossing" of two levels of the system. This is shown in fig. 9. The analytic continuation which the sum of ladders approximates could have been performed along any path within the region of convergence of the second summation. The real axis is excluded as the path of continuation to the left. In fact one must keep away from the real axis just far enough to go around one of the branch points B or B* and when one comes back to the real axis (for negative v) one is on the Riemann sheet corresponding to the normal state.

The ladder approximation should be viewed as an approximation to a certain expansion of the energy function. This expansion (which is in powers of the density) converges in a region in which the energy function is single-valued. To remain in this region one may pass any branch point only on one side and not on the other. As the path along the real axis is blocked by the pole one must go parallel to the real axis but far enough from it to avoid entering the region of non-convergence. As the branch points lie on the boundary of this region one must go parallel to the real axis but far enough from it to go around the branch points. This is precisely the path which yields everywhere the normal state.

The formal expression (5.8) which has been obtained by summing in the Brueckner order yields the normal branch of the energy function. The ladders (5.10) illustrated by the dashed line in fig. 9 approximate this branch well except in the vicinity of the pole.

6. The General Case

A general many-fermion system is of course far more complicated than the model of the last section. We shall attempt to show that the main features which have led us to conclude that the ladders describe the normal state of the system are the same. We limit ourselves to the low density limit (this is necessary in any case in order to retain only the first term of the second expansion) and use the Bethe-Goldstone equation [4]). This consists of assuming that particles interact only two at a time. Instead of a system of N particles one considers $\frac{1}{2}N(N-1)$ systems of two particles to which the outside of the Fermi sphere is available as well as two single particle states inside the sphere (fig. 10). The other states of the Fermi sphere are considered occupied by other particles.

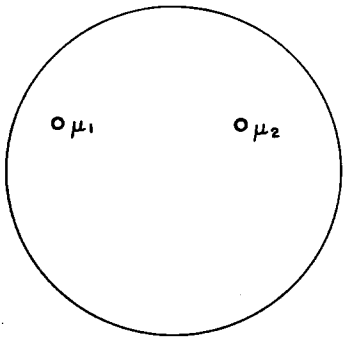

Fig. 10. The phase space available for the μ_1, μ_2 partial Bethe-Goldstone problem consists of the outside of the Fermi sphere and of the states μ_1 and μ_2.

The energy of the system is considered to be the sum of the energies of the two-particle systems:

$$E = \tfrac{1}{2} \sum_{\mu_1 \neq \mu_2} E_{\mu_1 \mu_2}, \tag{6.1}$$

where the states μ_1, μ_2 are inside the Fermi sphere.

This procedure implies the assumption that the wave function of the system never deviates considerably from the Fermi sphere. This assumption is justified for the normal state (while it may be quite wrong for the ground state). We defer the justification until the end of the section.

The $E_{\mu\mu'}$ are the energy functions of two-body problems restricted to Hilbert spaces and partial hamiltonians which are parts of the original ones. The following decomposition of the hamiltonian of the system is useful:

$$\begin{aligned} H &= \sum_{\lambda} \varepsilon_\lambda a_\lambda^\dagger a_\lambda + \tfrac{1}{4} \sum_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} v_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} a_{\lambda_2}^\dagger a_{\lambda_1}^\dagger a_{\lambda_3} a_{\lambda_4} \\ &= \tfrac{1}{2} \sum_{\mu \neq \mu'} (H_{0\mu\mu'} + v_{\mu\mu'}) + H'_0 + v' + v'' + v''', \end{aligned} \tag{6.2}$$

$$H_{0\mu\mu'} = \varepsilon_\mu a^\dagger_\mu a_\mu + \varepsilon_{\mu'} a^\dagger_{\mu'} a_{\mu'}, \tag{6.3}$$

$$v_{\mu\mu'} = \tfrac{1}{2} \sum_{\kappa\kappa'} (v_{\kappa\kappa'\,\mu\mu'} a^\dagger_{\kappa'} a^\dagger_\kappa a_\mu a_{\mu'} + v_{\mu\mu'\,\kappa\kappa'} a^\dagger_{\mu'} a^\dagger_\mu a_\kappa a_{\kappa'}), \tag{6.4}$$

$$H'_0 = \sum_\kappa \varepsilon_\kappa a^\dagger_\kappa a_\kappa, \tag{6.5}$$

$$v' = \tfrac{1}{4} \sum_{\kappa_1\kappa_2\kappa_3\kappa_4} v_{\kappa_1\kappa_2\kappa_3\kappa_4} a^\dagger_{\kappa_2} a^\dagger_{\kappa_1} a_{\kappa_3} a_{\kappa_4}, \tag{6.6}$$

$$v'' = \tfrac{1}{4} \sum_{\mu_1\mu_2\mu_3\mu_4} v_{\mu_1\mu_2\mu_3\mu_4} a^\dagger_{\mu_2} a^\dagger_{\mu_1} a_{\mu_3} a_{\mu_4}, \tag{6.7}$$

$$\begin{aligned} v''' = {} & \tfrac{1}{4} \sum_{\mu_1\mu_2\mu_3\kappa} (v_{\mu_1\mu_2\mu_3\kappa} a^\dagger_{\mu_2} a^\dagger_{\mu_1} a_{\mu_3} a_\kappa + v_{\mu_3\kappa\mu_2\mu_1} a^\dagger_\kappa a^\dagger_{\mu_3} a_{\mu_2} a_{\mu_1}) \\ & + \tfrac{1}{4} \sum_{\kappa_1\kappa_2\kappa_3\mu} (v_{\kappa_1\kappa_2\kappa_3\mu} a^\dagger_{\kappa_2} a^\dagger_{\kappa_1} a_{\kappa_3} a_\mu + v_{\kappa_3\mu\kappa_1\kappa_2} a^\dagger_\mu a^\dagger_{\kappa_3} a_{\kappa_1} a_{\kappa_2}). \end{aligned} \tag{6.7a}$$

The Greek indices denote the unperturbed single particle states (each index standing for the momentum, the spin is supressed). The summation on indices denoted by μ and κ are restricted to the inside and the outside of the Fermi sphere respectively. The summations on indices λ is unrestricted. The quantity ε_λ is the unperturbed single particle energy of the state λ (kinetic energy) and $v_{\lambda_1\lambda_2\lambda_3\lambda_4}$ are the matrix elements of the interaction between properly anti-symmetrized two-particle states:

$$v_{\lambda_1\lambda_2\lambda_3\lambda_4} \equiv (0|a_{\lambda_2} a_{\lambda_1} v a^\dagger_{\lambda_3} a^\dagger_{\lambda_4}|0). \tag{6.8}$$

We shall also define

$$H_{\mu\mu'} = H_{0\mu\mu'} + v_{\mu\mu'}, \tag{6.9}$$

$$H' = H'_0 + v', \tag{6.10}$$

where H' is the hamiltonian of particles restricted to the outside of the Ferm. sphere. The relevant hamiltonian for determining $E_{\mu\mu'}$ is $H_{\mu\mu'} + H'$. The part v'' of the interaction which corresponds to scattering within the Fermi sphere is not taken into account at all in the Bethe-Goldstone approximation. It is possible to generalize the procedure so as to include v'' in a way similar to v' but we are not concerned with this here. The part v''' also has no effect on the partial Bethe-Goldstone problems.

Let us now note that all the $\frac{1}{2}N(N-1)$ two-body problems have in common the unperturbed spectrum outside the Fermi sphere and the part of the hamiltonian H'. If any bound states arise as a result of the interaction they must be superpositions of large number of unperturbed states with amplitudes of the same order of magnitude. The contribution of any one unperturbed state to the formation of a bound state is thus not important. The bound states of the $\mu\mu'$ system will not be appreciably changed if we omit the μ and μ' states and consider only the states outside the Fermi sphere and the hamiltonian H'.

It follows that the bound states occurring in the different $\mu\mu'$ problems depend only on the total momentum $\mu+\mu'$. These bound states can be calculated considering a two-body problem in which the particles are restricted to the outside of the Fermi sphere and the hamiltonian is H'.

Not every attractive interaction gives rise to binding in the usual two-body problem. The nature of the ground state of a two-particle system is the result of a delicate balance between the kinetic and potential energy. While the potential energy is lowest if the particles stay close together within the range of attraction, this requires the use of a large number of unperturbed states. The

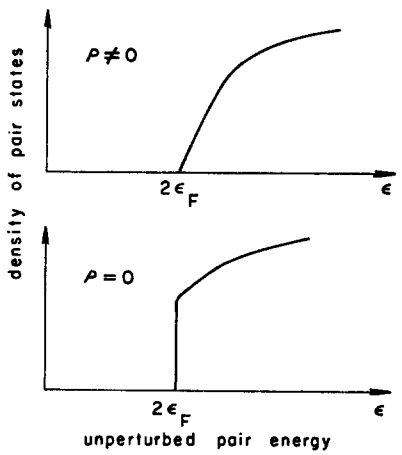

Fig. 11. The density of pair states of total momentum P (for $P \neq 0$ and $P = 0$) which lie outside the Fermi sphere.

kinetic energy is lowest if we use only those unperturbed states whose kinetic energy is low. The choice of such states is very limited in the usual two-body problem since the density of states is proportional to $\sqrt{\varepsilon}$, ε being the kinetic energy. The density of states of lowest energy ($\varepsilon = 0$) is zero. This scarcity of low lying unperturbed states prevents the formation of a bound state in many cases. Our problem is different. The low lying unperturbed states are not available anyway. The lowest kinetic energy in our considerations is the Fermi energy ε_F. States of this energy are available in finite density proportional to $\sqrt{\varepsilon_F}$. A more detailed examination of the situation shows that for pair states having total momentum different from zero the density of available unperturbed states starts from zero at $2\varepsilon_F$ (fig. 11), but it rises more rapidly the smaller the total momentum of the pair. For pairs of zero total momentum this rise is vertical and a finite density of states is available even at the lowest energy $2\varepsilon_F$. In this case (as has been shown by Cooper [2]) and more rigorously by Van Hove [9])) binding occurs for any attractive interaction.

Most of the explicit calculations on this subject have made use of separable interactions which, because of their special nature, give rise to a single bound

state. For a general interaction, however, there is no reason why several bound states cannot occur. The number of bound states will make no difference in principle in the following considerations. For the sake of simplicity we consider the case of a single bound state.

We therefore assume that for some total momenta, certainly for total momentum zero, there exists a bound state in the spectrum of H'. We denote its energy by $E_C^{\mu+\mu'}(g)$ (Cooper energy). As a consequence of the binding $E_C(g)$ is volume independent and $E_C(g) \to -\infty$ as $g \to -\infty$. This spectrum of H' is shown in fig. 12(a) for a certain total momentum of $\mu+\mu'$. Fig. 12(b) shows the spectrum of the $\mu\mu'$ partial problem. This spectrum has been obtained from

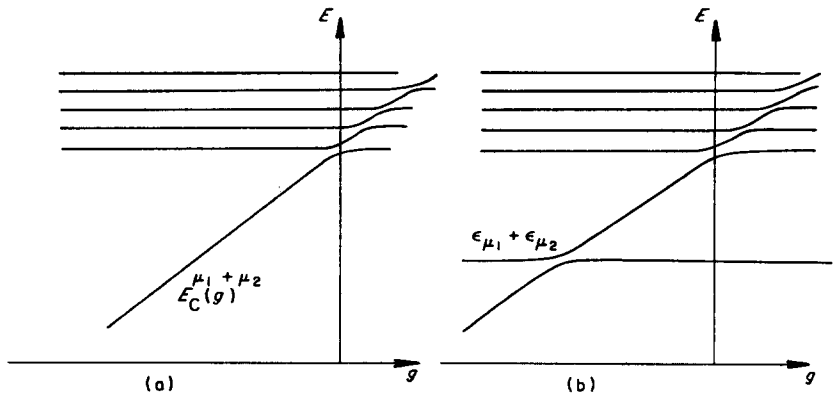

Fig. 12. (a) The spectrum of the two body problem making use only of the outside of the Fermi sphere, corresponding to total momentum $\mu_1+\mu_2$. (b) The spectrum of the partial μ_1, μ_2 Bethe-Goldstone problem.

fig. 12(a) by the addition of the level corresponding to the particles occupying the states μ and μ'. This level, which at $g = 0$ is lower than all the levels of H', is crossed or more exactly *almost* crossed by $E_C^{(P)}$ (g) for some value of $g^{(\mu\mu')}$. The energy function $E_{\mu\mu'}$ of the $\mu\mu'$ problem has two branch points on both sides of the real axis close to $g^{(\mu\mu')}$. We define $g^{(\mu\mu')}$ by

$$E_C^P(g^{(\mu\mu')}) = \varepsilon_\mu + \varepsilon_{\mu'}, \quad P = \mu+\mu'. \tag{6.11}$$

This equation has a solution at least when $\mu+\mu' = 0$. It is interesting to note that the smallest of the $g^{(\mu\mu')}$ gives the limit of the region of convergence of the perturbation expansion of E which is the sum of all the $E_{\mu\mu'}$. The smallest $g^{(\mu\mu')}$ corresponds to the problem in which $\varepsilon_\mu+\varepsilon_{\mu'}$ is closest to $2\varepsilon_F$, since for $g = 0$ $E_C(g)$ becomes $2\varepsilon_F$. The smallest $\varepsilon_\mu - \varepsilon_F$ is volume dependent (tending to zero as the volume tends to infinity), while E_C depends only on the density. It follows that the region of convergence of the perturbation expansion for a system of given density can be made as small as we wish by making the volume large enough.

Let us now turn to the diagram expansion. Summing these diagrams in the Brueckner order we first sum the particle-particle interactions to all orders and then proceed to sum hole-hole, creation and annihilation processes. This amounts to replacing the unperturbed propagator $1/(U-H_0)$ (U being the unperturbed energy) by

$$\frac{1}{U-H_0}+\frac{1}{U-H_0}v'\frac{1}{U-H_0}+\ldots=\frac{1}{U-H_0-v'}$$
$$=\frac{1}{U-\sum\varepsilon_m a_m^\dagger a_m-H'}=\frac{1}{\sum\varepsilon_m a_m a_m^\dagger-H'}. \quad (6.12)$$

The new propagator (6.12) includes implicitly all particle-particle interactions. It is immediately seen that when used in ladder diagrams the function (6.12) has a pole at each of the almost crossings $g^{(\mu\mu')}$. This follows from the fact that $E_C^{\mu+\mu'}(g)$ is an eigenvalue of H' while $\varepsilon_\mu+\varepsilon_{\mu'}$ is an eigenvalue of $\sum\varepsilon_\mu a_\mu a_\mu^\dagger$ for the case of two particles and two holes. These poles persist also in diagrams other than ladders, but in such diagrams other poles may appear as well. In the case of four particles and four holes a pole appears when

$$\varepsilon_{\mu_1}+\varepsilon_{\mu_2}+\varepsilon_{\mu_3}+\varepsilon_{\mu_4}+E_F^{P'}(g)+E_C^P(g)=0, \quad (6.13)$$

where $\mu_1+\mu_2+\mu_3+\mu_4=P'+P$ and $E_F^{P'}(g)$ is a free state of two particles with total momentum P'. Such poles appear at much larger values of g than the poles predicted by (6.11) because the binding of one pair has to compensate for the excitation of the other. Such poles, however, cancel when the different diagrams of the same order in explicit interactions are summed together. This must be so because each pole corresponds to a resonance in the scattering of two particles and therefore occurs when a bound state coincides in energy with one of the free pairs. The energies of the free pairs in question are always $\varepsilon_\mu+\varepsilon_{\mu'}$.

Let us see how this cancellation is expressed in diagrams. Let us consider the creation of two pairs of particles and holes, then two other pairs. Among the contributions to this process the following energy denominators will appear:

$$\frac{1}{\varepsilon_{\mu_1}+\varepsilon_{\mu_2}-E_F^{\mu_1+\mu_2}}\quad\frac{1}{\varepsilon_{\mu_1}+\varepsilon_{\mu_2}+\varepsilon_{\mu_3}+\varepsilon_{\mu_4}-E_F^{\mu_1+\mu_2}-E_C^{\mu_3+\mu_4}},$$
$$\frac{1}{\varepsilon_{\mu_3}+\varepsilon_{\mu_4}-E_C^{\mu_3+\mu_4}}\quad\frac{1}{\varepsilon_{\mu_1}+\varepsilon_{\mu_2}+\varepsilon_{\mu_3}+\varepsilon_{\mu_4}-E_F^{\mu_1+\mu_2}-E_C^{\mu_3+\mu_4}}. \quad (6.14)$$

The corresponding creation and annihilation matrix elements are the same. Summing the two contributions in (6.14) together we have

$$\frac{1}{\varepsilon_{\mu_1}+\varepsilon_{\mu_2}-E_F^{\mu_1+\mu_2}}\quad\frac{1}{\varepsilon_{\mu_3}+\varepsilon_{\mu_4}-E_C^{\mu_3+\mu_4}}, \quad (6.15)$$

and the spurious pole has disappeared. This proof is exact in diagrams corresponding to partial Bethe-Goldstone problems. We hope to be able to give a general proof in another publication.

We therefore conclude that each term in the second summation has poles at the points $g^{(\mu\mu')}$.

Fig. 13 summarizes the analytic properties of the energy function E of the system and of the summation of ladders. The quantity E is the sum of the $E_{\mu\mu'}$, and therefore has branch points close to the points $g^{(\mu\mu')}$ of the real axis. The new propagator (6.12) has poles at all the points $g^{(\mu\mu')}$. The points $g^{(\mu\mu')}$ are denoted by P_i in the drawing and the branch points by B_i, B_i*. The situation shown in fig. 13 is completely analogous to the one shown in fig. 8 for the case

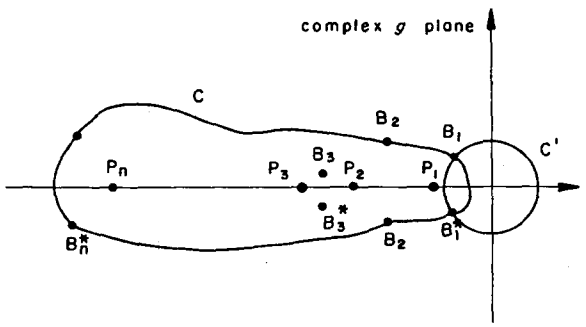

Fig. 13. The analytic properties of the energy function in the complex g plane. The perturbation expansion converges inside C' and the second summation outside C.

of the model of sect. 5. The usual perturbation expansion converges only within a circle around the origin which passes through B_1 and B_1*, the nearest singularities. The second summation in the Brueckner case is a summation of the contributions of diagrams in which there is no explicit particle-particle scattering and making use of the propagator (6.12). This propagator is infinite at the points P_1—P_n of the real axis and therefore the second summation certainly fails to converge there. The terms of the second summation depend on g explicitly and also implicitly through the propagator. We shall again assume that the implicit dependence is the most important one in the neighbourhood of the points where the propagator is infinite. This assumption again turns the second summation into an expansion in powers of the propagator. Such an expansion converges far away from the segment P_1—P_n, the singularities B_i, B_i* lying soutside the region of convergence or on its boundaries. The boundary of the region of convergence is indicated by the curve C in fig. 13. The second summation converges everywhere outside C, and there it may be approximated by the sum of the first few terms or even of the ladders alone. The second summation represents an analytic continuation of the energy function along any path within its region of convergence. All these paths corre-

spond to the normal state since they lie off the real axis and farther away from it than the branch points.

It should be noted that the argument we have used does not imply that the second summation actually converges everywhere outside the curve C. Singularities arising from the explicit dependence on g of the terms of the second summation and corresponding to branch points connecting the different unbound states may limit the region of convergence in the far away parts of the g plane. We only claim that in the neighbourhood of the segment P_1-P_n and the curve C our description is correct. The Brueckner procedure therefore corresponds to analytic continuation along a path lying off the real axis, but just far enough from it to go around all the branch points corresponding to binding of pairs and to lead to the normal state.

It still remains for us to justify the Bethe-Goldstone assumption that the distribution of particles over states in the normal state of the system deviates only little from the Fermi sphere. This we shall do by induction. Suppose the assumption is true for the normal state at a value of the coupling constant g just before the nth almost crossing. Analytic continuation along the real axis would lead to a state in which the corresponding pair is bound and therefore stays outside the Fermi sphere. But we do the continuation around the branch point and come back to the state of the system where this pair, as all others, is unbound and the assumption is thus satisfied also *after* the nth almost crossing. Since the assumption is satisfied for the normal state $g = 0$, i.e. before the first almost crossing, it is satisfied after any number of almost crossings. Therefore there occurs no binding in the normal state for any value of g. This establishes the physical meaning of the normal state as the lowest state of the system in which no binding occurs.

I wish to thank Professor Harry J. Lipkin for many fruitful discussions.

References

1) C. Bloch, Nuclear Physics **7** (1958) 451
2) L. N. Cooper, Bull. Am. Phys. Soc. **6** (1961) 16 and preprint
3) K. A. Brueckner, in the many body problem (Dunod, Paris, 1959)
4) H. A. Bethe and J. Goldstone, Proc. Roy Soc. **238** (1957) 551
5) A. Katz, Nuclear Physics **18** (1960) 177
6) L. N. Cooper, Phys. Rev. **104** (1956) 1189
7) J. M. Blatt, Prog. Theor. Phys. **23** (1960) 447
8) A. Katz, Nuclear Physics **20** (1960) 663
9) L. Van Hove, Physica **25** (1959) 849