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PT-symmetric quantum theory

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$\mathcal{PT}$-symmetric quantum theory

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Abstract. The average quantum physicist on the street would say that a quantum-mechanical Hamiltonian must be Dirac Hermitian (invariant under combined matrix transposition and complex conjugation) in order to guarantee that the energy eigenvalues are real and that time evolution is unitary. However, the Hamiltonian $H = p^2 + ix^3$, which is obviously not Dirac Hermitian, has a positive real discrete spectrum and generates unitary time evolution, and thus it defines a fully consistent and physical quantum theory. Evidently, the axiom of Dirac Hermiticity is too restrictive. While $H = p^2 + ix^3$ is not Dirac Hermitian, it is $\mathcal{PT}$ symmetric; that is, invariant under combined parity $\mathcal{P}$ (space reflection) and time reversal $\mathcal{T}$. The quantum mechanics defined by a $\mathcal{PT}$-symmetric Hamiltonian is a complex generalization of ordinary quantum mechanics. When quantum mechanics is extended into the complex domain, new kinds of theories having strange and remarkable properties emerge. In the past few years, some of these properties have been verified in laboratory experiments. A particularly interesting $\mathcal{PT}$-symmetric Hamiltonian is $H = p^2 - x^4$, which contains an upside-down potential. This potential is discussed in detail, and it is explained in intuitive as well as in rigorous terms why the energy levels of this potential are real, positive, and discrete. Applications of $\mathcal{PT}$-symmetry in quantum field theory are also discussed.

1. Introduction

Physical systems whether they are classical or quantum-mechanical, are described by a quantity called the *Hamiltonian*. In conventional quantum physics two kinds of Hamiltonians are used, (i) Hermitian Hamiltonians, which govern the behavior of isolated systems, and (ii) non-Hermitian Hamiltonians, which govern the behavior of systems in contact with the environment. Hermitian Hamiltonians describe idealized systems in equilibrium whose total energy and probability are conserved; the energy levels of such systems are real. Non-Hermitian Hamiltonians describe the phenomenological behavior of experimental systems involved in scattering or decay processes. Such systems receive energy from and/or deposit energy into their environment, so they are typically not in equilibrium, the total energy and probability are not conserved, and the energy levels are complex numbers; the imaginary part of an energy level is associated with the lifetime of the physical state or the width of a scattering resonance.

This talk introduces and summarizes new kinds of Hamiltonians called $\mathcal{PT}$*-symmetric* Hamiltonians, which are intermediate between Hermitian and non-Hermitian Hamiltonians. Like non-Hermitian systems, $\mathcal{PT}$-symmetric systems are not isolated, but their contact with the environment is highly constrained so that gain from the environment and loss to the environment is exactly balanced. As a consequence, even though they are not isolated, $\mathcal{PT}$-symmetric systems behave like Hermitian systems in that they are in equilibrium and their energy levels are real.

The special kinds of systems discussed here are called $\mathcal{PT}$ symmetric because they are invariant under the combined operation of space reflection (designated parity $\mathcal{P}$) and reversing the arrow of time (designated $\mathcal{T}$). $\mathcal{PT}$-symmetric quantum systems were discovered in 1998 (in a highly theoretical and abstract context) by Prof. Bender [1], who proposed the idea of extending real (Hermitian) quantum theory into the complex (non-Hermitian) domain. (By analogy, in mathematics it is extremely useful to extend the real numbers to the complex numbers; doing so helps to elucidate the properties of the real number system.) Since its inception sixteen years ago there have been dozens of experimental confirmations published in *Nature*, *Science*, and *Physical Review Letters* in diverse research areas such as optics [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14], NMR [15], superconductivity [16, 17], microwave cavities [18, 19], lasers [20, 21], atomic diffusion [22], electronic circuits [23, 24], chaos and noise [25, 26], mechanical systems [27], and graphene [28, 29]. Some particularly interesting recent experimental work involves the study of $\mathcal{PT}$-symmetric whispering-gallery microcavities [30, 31, 32]. New experiments involving Bose-Einstein condensates are in the planning stage [33, 34, 35] and new $\mathcal{PT}$-symmetric metamaterials are being developed [36, 37, 38, 39]. As an indication that the study of $\mathcal{PT}$ symmetry is now part of the scientific and technological mainstream, it is noteworthy that in August 2012 the U.S. Air Force Office of Scientific Research issued a Broad Agency Announcement (BAA) calling for scientific proposals, including research and development of $\mathcal{PT}$-symmetric optical systems and synthetic $\mathcal{PT}$ metamaterials [40]. (A \$5 million experimental MURI grant was awarded in June 2013.) This BAA is a significant recognition of $\mathcal{PT}$ symmetry, which was originated by Prof. Bender just 14 years earlier.

Over the past sixteen years, the research area of $\mathcal{PT}$ symmetry has become highly interdisciplinary with an explosion of theoretical and experimental research and new papers appearing on the arXiv daily. There are now nearly two thousand published papers on the subject of $\mathcal{PT}$ symmetry. This explosive growth is reflected in the fact that there have now been more than twenty international conferences entirely devoted to $\mathcal{PT}$-symmetric quantum theory.

2. Elementary $\mathcal{PT}$-symmetric quantum-mechanical systems

An example of a class of non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians having entirely real, positive, and discrete spectra is

$$H = p^2 + x^2(ix)^\varepsilon \quad (\varepsilon \text{ real}) \tag{1}$$

(see Fig. 1). The spectrum of H for $\varepsilon > 0$ is real because its $\mathcal{PT}$ symmetry is *unbroken* (that is, the eigenstates of H are also eigenstates of $\mathcal{PT}$). When $\varepsilon < 0$, the $\mathcal{PT}$ symmetry of H is broken and the spectrum is not real. Conventional Hermitian quantum mechanics (here, the harmonic oscillator) sits at the transition between the regions of unbroken and broken $\mathcal{PT}$ symmetry. Ref. [1] proposed that the physical requirement of $\mathcal{PT}$ symmetry (that is, spacetime reflection symmetry) could be used in place of the mathematical condition of Hermiticity. Because $\mathcal{PT}$ symmetry is a weaker requirement than Hermiticity, Hamiltonians that previously would have been rejected as unphysical can now be considered as potentially valid descriptions of physical processes.

A reader unfamiliar with $\mathcal{PT}$-symmetric quantum mechanics will regard Fig. 1 as astonishing because it shows (contrary to what one would naively expect) that the eigenvalues of $H = p^2 + ix^3$ (pure imaginary potential) and $H = p^2 - x^4$ (upside-down potential) are *real*, *positive*, and *discrete*. An intuitive explanation of why the eigenvalues of the upside-down $-x^4$ potential are real, positive, and discrete was given by Bender along with Berry and Ahmed [41] and an exact calculation can be found in Ref. [42]. Dorey, Dunning, and Tateo proved *rigorously* that for all $\varepsilon \geq 0$ the eigenvalues of $H = p^2 + x^2(ix)^\varepsilon$ are real [43, 44]. Their proof establishes a beautiful mathematical correspondence between ordinary-differential-equation eigenvalue problems and integrable models, which is known as the *ODE/IM correspondence* [45].

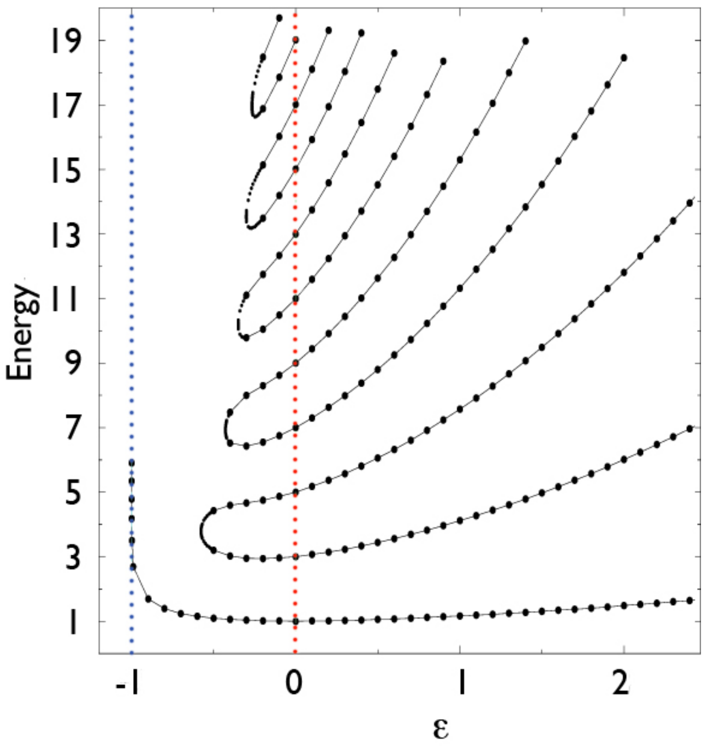

Figure 1. Energy levels of the parametric family of Hamiltonians $H = p^2 + x^2(ix)^\varepsilon$ in (1). When $\varepsilon \geq 0$, the eigenvalues are all real and positive, and they increase with increasing ε. When ε decreases below 0, the eigenvalues disappear into the complex plane as complex conjugate pairs. Eventually, only one real eigenvalue remains when ε is less than about -0.57, and as ε approaches -1 from above, this eigenvalue becomes infinite.

3. The $\mathcal{PT}$ phase transition

A crucial aspect of the energy levels in Fig. 1 is that they exhibit a transition, known as the $\mathcal{PT}$ *phase transition*, in which they become pairwise degenerate at exceptional points and move off into the complex plane. This transition is a generic characteristic of $\mathcal{PT}$-symmetric systems but cannot occur in conventional Hermitian quantum systems because such systems can never have complex energies. This phase transition is the signature and characteristic behavior that was observed in many of the experiments mentioned above.

4. Extending physical theories into the complex plane tames instabilities

The discovery of complex numbers was a major development in mathematics because it allowed one to solve equations such as $1 = -x^4$ that were thought to have no solution. Analogously, extending physics into the complex plane also yields new insights. For example, the conventional belief is that one should reject theories based on apparently unstable upside-down potentials such as $V(x) = -x^4$ because a classical particle on the real axis subject to this potential will zoom off to $\pm\infty$ (unless it is precariously balanced at rest at $x = 0$). As a consequence, one would believe that the spectrum of the Hamiltonian for the quantum system is not bounded below and that there is no ground state. This belief is wrong!

How does an upside-down quartic potential confine classical and quantum particles? Let us calculate the time for a classical particle of energy E in an $-x^4$ potential to travel to infinity. The time of flight T from $x = 0$ to $x = \infty$ for a particle described by the Hamiltonian $H = p^2 - x^4$ is

$$T = \int_{t=0}^{T} dt = \int_{x=0}^{\infty} \frac{dx}{\dot{x}} = \int_{x=0}^{\infty} \frac{dx}{2\sqrt{E + x^4}}.$$

This time is *finite* because the integral exists. After the particle reaches infinity in finite time, where does it go next? To answer this question we examine the particle's motion in the complex-x plane. Figure 2 shows four trajectories in the upper-half x plane; the trajectories are all *closed*, *periodic*, and *stable* orbits. (One trajectory represents oscillatory motion between the turning points at $x = e^{i\pi/4}$ and $x = e^{3i\pi/4}$.) We emphasize that all trajectories of a particle of positive energy E in a $-x^4$ potential are *stable* $\mathcal{PT}$-symmetric (left-right symmetric) orbits encircling pairs of turning points, which are the solutions to the equation

$$E = -x^4.$$

The unstable motion on the real-x axis becomes infinitesimally rare (with probabilistic measure zero). The classical particles spend almost all their time near the origin $x = 0$ because this is where the particles are moving slowest, so $x = 0$ is *NOT* a point of instability. As the trajectories approach the real axis, the classical particle does not simply disappear at $x = \pm\infty$. Rather, as the particle reaches $\pm\infty$, it instantly reappears at $\mp\infty$. Thus, the classical particle *periodically* completes the transit from $\pm\infty$ to $\mp\infty$. (This periodic motion is equivalent to a flux of particles on the real axis with a source at $\pm\infty$ and a sink at $\mp\infty$.) A three-dimensional plot of the absolute value of the complex classical probability is shown in Fig. 3.

If we quantize the theory, we find that an upside-down potential has discrete quantum bound states. Indeed, the $-x^4$ potential has bound states that are *localized* at $x = 0$. This counterintuitive, but rigorous fact contradicts the naive view that the origin is unstable. To see why the quantum energies are discrete and not continuous, recall the Bohr-Sommerfeld concept of quantization. A quantum particle is a wave, and if the classical motion is periodic, the wave must interfere with itself constructively. The condition for constructive interference is

$$\oint dx \sqrt{E - V(x)} = (n + 1/2)\pi \quad (n = 0,\, 1,\, 2,\, \ldots),$$

which is the complex version of the WKB quantization condition. Thus, complex classical mechanics provides the insight to explain how complex quantum-mechanical systems work and why complex systems can be stable when their real counterparts appear to be unstable.

5. Differences between $\mathcal{PT}$-symmetric and conventional quantum theory

$\mathcal{PT}$-symmetric quantum theory is not in conflict with conventional quantum theory; rather, it is generalization of it. An important early discovery regarding $\mathcal{PT}$-symmetric quantum theory is that there is a (positive-metric) Hilbert space if the $\mathcal{PT}$ symmetry of the quantum theory is unbroken (which means that the energy spectrum is real and bounded below) [46]. This *PRL* showed that when the $\mathcal{PT}$ symmetry is unbroken, the theory possesses a hidden symmetry represented by a linear operator $\mathcal{C}$, which commutes with the Hamiltonian. $\mathcal{C}$ represents a new *observable quantum number* mathematically related to particle-antiparticle symmetry. The Hilbert-space metric is the $\mathcal{CPT}$ inner product, and with respect to this new inner product (rather than the conventional Dirac inner product) the non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian generates unitary time evolution. Thus, *a $\mathcal{PT}$-symmetric theory is consistent with the fundamental axioms of quantum mechanics.* These results were extended from $\mathcal{PT}$-symmetric quantum mechanics to $\mathcal{PT}$-symmetric (spacetime symmetric) quantum field theory in a subsequent *PRL* [47] and in a *PRD* paper [48]. (For recent work on the $\mathcal{C}$ operator, see Refs. [49, 50, 51].)

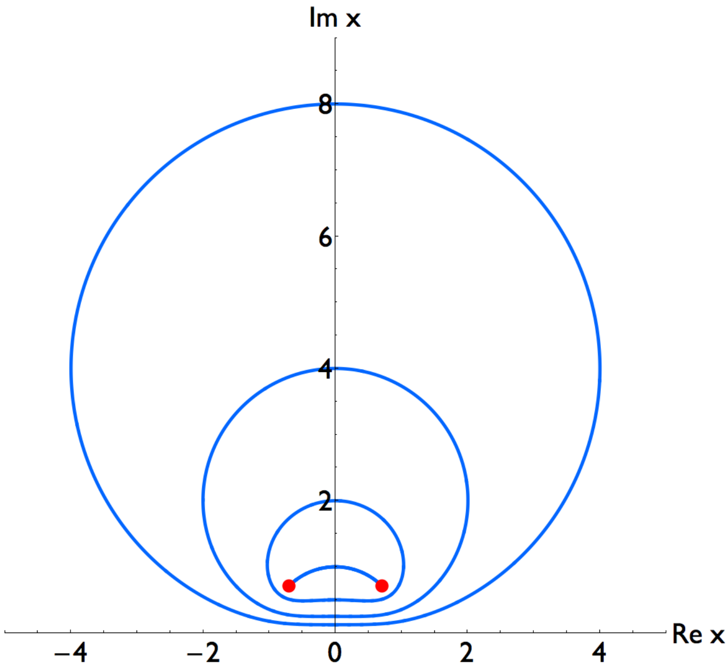

Figure 2. Closed periodic trajectories for $H = p^2 - x^4$ in the upper-half complex-x plane. The energy is $E = 1$. Turning points located at $x = e^{i\pi/4}$ and $x = e^{3i\pi/4}$ are indicated by dots. (The two other turning points in the lower-half plane are not shown.) Initial conditions are $x_0 = i,\ i/2,\ i/4,\ i/8$. A key feature of $\mathcal{PT}$-symmetric systems is that while they appear open from the narrow perspective of the real axis, they become closed when extended to the complex plane.

To see that $\mathcal{PT}$ symmetry is a generalization of standard Hermiticity, we take the limit $\varepsilon \to 0$ in the Hamiltonian $H = p^2 + x^2(ix)^\varepsilon$. In this limit the $\mathcal{CPT}$ norm reduces to the Hermitian conjugate norm of conventional quantum theory. The connection between the $\mathcal{C}$ operator in $\mathcal{PT}$-symmetric quantum field theory and the charge-conjugation operator in conventional quantum field theory is not yet understood. They are both Lorentz scalars [52], but in particle physics parity $\mathcal{P}$ and charge conjugation $\mathcal{C}$ commute; in $\mathcal{PT}$-symmetric quantum theory they do not. Thus, $\mathcal{PT}$-symmetric "particles" (states of positive $\mathcal{C}$) and "antiparticles" (states of negative $\mathcal{C}$) may have different masses. It is conceivable that this fact might account for the so-far-unexplained baryon-antibaryon asymmetry of the universe.

$\mathcal{PT}$-symmetric quantum systems have characteristically different behaviors from ordinary quantum systems. In addition to the $\mathcal{PT}$ phase transition, $\mathcal{PT}$-symmetric systems can evolve *faster* than conventional Hermitian quantum-mechanical systems, as claimed in Ref. [53]. The predicted fast time evolution has been observed in NMR experiments [15]. Also, DeKieviet, an experimentalist at the University of Heidelberg (probably the world's expert on atomic-beam spin-echo experiments) observed this predicted fast time evolution in experiments involving He^3 beams. He presented his work at an ECT conference in 2012 [54]. Another characteristic difference: Hermitian periodic potentials, such as $V(x) = \sin x$, exhibit energy bands and gaps. At one edge of each band of allowed energies the wave function is bosonic (2π periodic); at the other edge the wave function is fermionic (4π periodic). However, while $\mathcal{PT}$-symmetric periodic potentials such as $V(x) = i \sin x$ also have real energy bands and gaps, there are half as many gaps [55]. At the band edges the wave function is always bosonic and never fermionic. It would

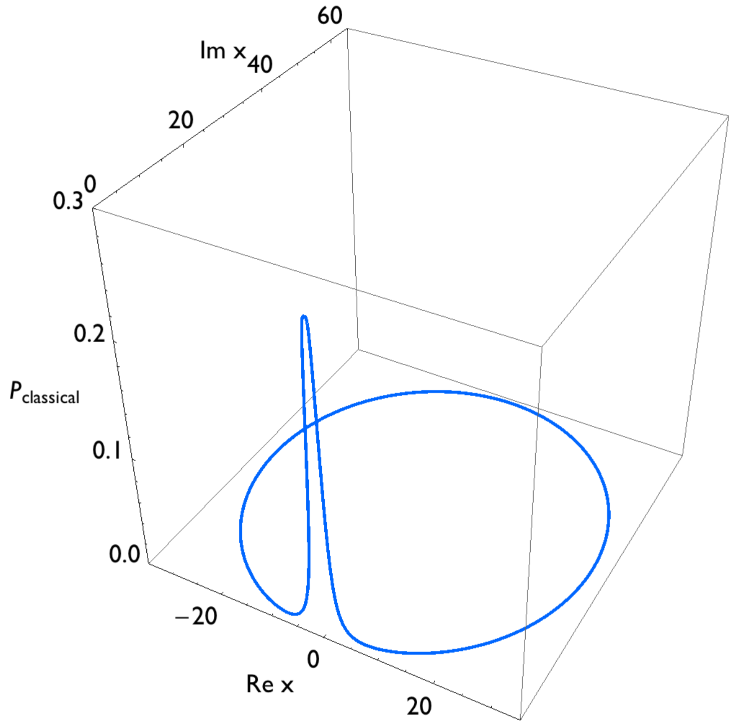

Figure 3. Three-dimensional plot of the probability (inverse of the absolute value of $\dot{x}$) for a contour beginning at $x_0 = i/64$ in the complex plane. The classical energy is $E = 1$. The classical particle executes a rapid loop in the complex plane but goes slowly near the origin.

be a difficult experiment (possibly involving neutron scattering), but a laboratory observation of a material having this kind of band structure would be clear evidence of a fundamental physical system described by a $\mathcal{PT}$-symmetric Hamiltonian; such a material could be regarded as a complex $\mathcal{PT}$-symmetric crystal.

6. Illustrative 2×2 matrix example of a $\mathcal{PT}$-symmetric Hamiltonian

Let us consider the elementary 2×2 Hamiltonian matrix

$$H = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix},$$

where the three parameters r, s, and θ are real. (This matrix Hamiltonian was discussed in Ref. [56].) The Hamiltonian H is, of course, not Hermitian, but it is easy to see that it is $\mathcal{PT}$ symmetric, where we define the parity operator as

$$\mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and we define the operator $\mathcal{T}$ to perform complex conjugation.

As a first step in analyzing the Hamiltonian H, we calculate its two eigenvalues:

$$E_\pm = r\cos\theta \pm (s^2 - r^2\sin^2\theta)^{1/2}.$$

There are clearly two parametric regions to consider, one for which the square root is real and the other for which it is imaginary. When $s^2 < r^2\sin^2\theta$, the energy eigenvalues form a complex conjugate pair. This is the region of broken $\mathcal{PT}$ symmetry. On the other hand, when $s^2 \geq r^2 \sin^2\theta$, the eigenvalues are real. This is the region of unbroken $\mathcal{PT}$ symmetry. In the unbroken region the simultaneous eigenstates of the operators H and $\mathcal{PT}$ are

$$|E_+\rangle = \frac{1}{\sqrt{2\cos\alpha}}\begin{pmatrix} e^{i\alpha/2} \\ e^{-i\alpha/2}\end{pmatrix} \quad \text{and} \quad |E_-\rangle = \frac{i}{\sqrt{2\cos\alpha}}\begin{pmatrix} e^{-i\alpha/2} \\ -e^{i\alpha/2}\end{pmatrix},$$

where

$$\sin\alpha = \frac{r}{s}\sin\theta.$$

The $\mathcal{PT}$ inner product (which is *not* the correct inner product) gives

$$(E_\pm, E_\pm) = \pm 1 \quad \text{and} \quad (E_\pm, E_\mp) = 0,$$

where $(u, v) = (\mathcal{PT}u)\cdot v$. With respect to the $\mathcal{PT}$ inner product, the vector space spanned by the energy eigenstates has a metric of signature $(+,-)$. If the condition $s^2 > r^2\sin^2\theta$ for an unbroken $\mathcal{PT}$ symmetry is violated, the states are no longer eigenstates of $\mathcal{PT}$ because α becomes imaginary. When $\mathcal{PT}$ symmetry is broken, the $\mathcal{PT}$ norm of the energy eigenstate vanishes.

Next, we construct the operator $\mathcal{C}$:

$$\mathcal{C} = \frac{1}{\cos\alpha}\begin{pmatrix} i\sin\alpha & 1 \\ 1 & -i\sin\alpha\end{pmatrix}.$$

Note that $\mathcal{C}$ is distinct from H and $\mathcal{P}$ and it has the key property that

$$\mathcal{C}|E_\pm\rangle = \pm|E_\pm\rangle.$$

The operator $\mathcal{C}$ commutes with H and satisfies $\mathcal{C}^2 = 1$. The eigenvalues of $\mathcal{C}$ are precisely the signs of the $\mathcal{PT}$ norms of the corresponding eigenstates. Using the operator $\mathcal{C}$ we construct the new inner product structure $\langle u|v\rangle = (\mathcal{CPT}u)\cdot v$. This inner product is positive definite because $\langle E_\pm|E_\pm\rangle = 1$. Thus, the two-dimensional Hilbert space spanned by $|E_\pm\rangle$, with inner product $\langle\cdot|\cdot\rangle$, has signature $(+,+)$.

Finally, we show that the $\mathcal{CPT}$ norm of any vector is positive. For the arbitrary vector $\psi = \binom{a}{b}$, where a and b are any complex numbers, we see that

$$\begin{aligned}
\mathcal{T}\psi &= \begin{pmatrix} a^* \\ b^*\end{pmatrix}, \\
\mathcal{PT}\psi &= \begin{pmatrix} b^* \\ a^*\end{pmatrix}, \\
\mathcal{CPT}\psi &= \frac{1}{\cos\alpha}\begin{pmatrix} a^* + ib^*\sin\alpha \\ b^* - ia^*\sin\alpha\end{pmatrix}.
\end{aligned}$$

Thus,

$$\langle\psi|\psi\rangle = (\mathcal{CPT}\psi)\cdot\psi = \frac{1}{\cos\alpha}[a^*a + b^*b + i(b^*b - a^*a)\sin\alpha].$$

Now let $a = x + iy$ and $b = u + iv$, where x, y, u, and v are real. Then

$$\langle\psi|\psi\rangle = \frac{1}{\cos\alpha}\left(x^2 + v^2 + 2xv\sin\alpha + y^2 + u^2 - 2yu\sin\alpha\right),$$

which is explicitly positive and vanishes only if $x = y = u = v = 0$.

Since $\langle u|$ denotes the $\mathcal{CPT}$-conjugate of $|u\rangle$, the completeness condition reads

$$|E_+\rangle\langle E_+| + |E_-\rangle\langle E_-| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Furthermore, using the $\mathcal{CPT}$ conjugate $\langle E_\pm|$, we can represent $\mathcal{C}$ as

$$\mathcal{C} = |E_+\rangle\langle E_+| - |E_-\rangle\langle E_-|.$$

In the limit $\theta \to 0$, the matrix Hamiltonian H for this two-state system becomes Hermitian and $\mathcal{C}$ reduces to the parity operator $\mathcal{P}$. Thus, $\mathcal{CPT}$ invariance reduces to the standard condition of Hermiticity for a symmetric matrix; namely, $H = H^*$.

7. $\mathcal{PT}$ symmetry clarifies questions in conventional quantum mechanics and quantum field theory

$\mathcal{PT}$ symmetry has explained some long-standing mysteries regarding non-Hermitian Hamiltonians that were used in the past. For example, in 1959 Wu showed that the ground state of a Bose system of hard spheres is described by a non-Hermitian Hamiltonian [57]. Wu found that the ground-state energy is real and conjectured that all energy levels were real. In 1992 Hollowood found that while the Hamiltonian for a complex Toda lattice is non-Hermitian, the energy levels are real [58]. Non-Hermitian Hamiltonians of the form $H = p^2 + ix^3$ and cubic quantum field theories having imaginary self-interaction terms also arise in studies of the Lee-Yang edge singularity [59, 60, 61, 62] and in various Reggeon field-theory models [63, 64, 65]. In all these cases a non-Hermitian Hamiltonian having a real spectrum was confusing at the time; the explanation is now clear: In each case the non-Hermitian Hamiltonian is $\mathcal{PT}$-symmetric.

8. $\mathcal{PT}$ symmetry tames instabilities in quantum field theory

When a quantum field theory has *ghosts* (negative-norm states) that arise when renormalizing the field theory, it is because the Hamiltonian has become non-Hermitian as a consequence of renormalization. Often, as in the case of the Lee Model, this Hamiltonian is $\mathcal{PT}$-symmetric, and if it is quantized using the $\mathcal{CPT}$ metric appropriate for the Hamiltonian instead of the standard Dirac inner product, these "ghost states" can be re-interpreted as conventional physical states having positive norm. A case in point is the Lee model, which was proposed by T. D. Lee in 1954 as a quantum field theory in which the renormalization program could be carried out exactly and in closed form [66]. One year later, Källén and Pauli found that upon renormalization the model develops a new negative-norm state that makes the S matrix nonunitary [67]. The problem of how to treat this ghost state was studied by well-known physicists, including Pauli, Heisenberg, and Wick, but it remained unsolved for 50 years. The prevailing belief was that the Lee model was fundamentally flawed [68, 69]. However, in 2005 this 50-year-old problem solved [70]. This paper showed that the renormalized Lee-model Hamiltonian is $\mathcal{PT}$-symmetric. The $\mathcal{C}$ operator was calculated exactly, the "ghost" state was shown to be a physically acceptable state, and that the S matrix was demonstrated to be unitary. Smilga *et al.* [71] and Curtright *et al.* [72, 73] followed this approach and showed that other quantum theories that were thought to contain ghost states actually have physically acceptable positive-norm states when the theory is quantized according to the techniques of $\mathcal{PT}$-symmetric quantum theory.

9. $\mathcal{PT}$ symmetry tames instabilities in the Pais-Uhlenbeck model

In 2008 it was shown that $\mathcal{PT}$ symmetry also eliminates the ghost states in the Pais-Uhlenbeck Model, which is a model higher-derivative quantum field theory. The solution to this 50-year-old problem was given in Refs. [74, 75]. Again, the Hamiltonian for this model is not Hermitian but it is $\mathcal{PT}$-symmetric, and the long-standing belief that this model possesses states of negative norm is false. These papers show that there may not be a problem with quantizing higher-derivative field theories. Indeed, the "ghost" in Pauli-Villars regularization may not be a ghost at all, but rather an indication that the wrong Hilbert-space metric is being used.

10. $\mathcal{PT}$ symmetry tames instabilities in the double-scaling limit

The conventional double-scaling limit of an O(N)-symmetric quartic quantum field theory is inconsistent because the critical coupling constant is negative. Thus, the partition function integral for the critical theory does not exist. Bender, Moshe, and Sarkar showed that $\mathcal{PT}$-symmetric quantum theory resolves this problem. In a first paper they demonstrated that a zero-dimensional O(N)-symmetric quantum field theory avoids this difficulty if the original quartic theory is replaced by its $\mathcal{PT}$-symmetric analog [76]. A second paper on one-dimensional O(N)-symmetric quartic quantum field theory [O(N)-symmetric quantum mechanics] showed that the $\mathcal{PT}$-symmetric formulation of the Schrödinger equation provides a consistent double-scaling limit [77].

11. Stable $\mathcal{PT}$-symmetric timelike logarithmic Liouville quantum field theory

Complex Liouville quantum field theory has a self-interaction term of the form $e^{i\phi}$ and is a topic of great interest at both the experimental and the theoretical level [78, 79, 80, 81, 82, 83]. A quantum field theory having such a self-interaction term is $\mathcal{PT}$ symmetric. However, it is a remarkable fact that the seemingly unstable $\mathcal{PT}$-symmetric self-interaction term of the form $i\phi e^{i\phi}$ also gives a theory with a stable vacuum. In fact, this interaction leads to an infinite number of quantum field theories each of which has a stable vacuum [84].

12. Possible applications of $\mathcal{PT}$-symmetry in the Higgs sector of the SM

There are a number of interesting possible applications of $\mathcal{PT}$-symmetric quantum theory to the Higgs sector of the Standard Model. In the conventional formulation the Higgs field has a quartic ϕ^4 interaction term. However, in four dimensions a ϕ^4 quantum field theory is not asymptotically free and the theory is trivial (noninteracting). In contrast, a $\mathcal{PT}$-symmetric $-\phi^4$ theory is asymptotically free and nontrivial. Furthermore, the vacuum expectation value $\langle\phi\rangle$ is nonzero, not because of spontaneous symmetry breaking, but simply because the boundary conditions on the partition functional integral automatically imply that $\langle\phi\rangle \neq 0$. Of course, in the past a $-\phi^4$ model for the Higgs particle was not considered because it was assumed that the energy spectrum would be unbounded below. In fact, this is not a problem; a $-\phi^4$ theory is stable and has a positive spectrum.

There are further problems with the Higgs model. If one renormalizes the model, one generates a new interaction term of the form $-\phi^4 \log\phi$ in addition to the original ϕ^4 interaction term, which suggests that the Higgs vacuum becomes unstable [85]. One possible interpretation of this result is to concede that the vacuum really is unstable but that the lifetime of the vacuum is sufficiently long that one need not be concerned. We believe that this is a rather unappealing interpretation. However, as we have repeatedly pointed out, by extending this quartic interation into the complex domain and treating the interaction term by using the methods of $\mathcal{PT}$-symmetric quantum theory, we believe that the vacuum actually becomes stable.

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