Quantum brachistochrone problem for a spin-1 system in a magnetic field

A. M. Frydryszak[1,*] and V. M. Tkachuk[2,†]
[1]*Institute of Theoretical Physics, University of Wroclaw, place M. Borna 9, 50-204 Wroclaw, Poland*
[2]*Department of Theoretical Physics, Ivan Franko Lviv National University, 12 Drahomanov Street, Lviv UA-79005, Ukraine*
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We study the quantum brachistochrone problem for a spin-1 system in a magnetic field of constant absolute value. Such a system gives us the possibility to examine in detail the statement that the state vectors realizing evolution with the minimal time of passage evolve along the subspace spanned by the initial and final state vectors [Carlini *et al.*, Phys. Rev. Lett. **96**, 060503 (2006); Brody and Hook, J. Phys. A **39**, L167 (2006)]. Considering an explicit example, we show the existence of a quantum brachistochrone with the minimal possible time; however, the state vector we study leaves the subspace spanned by the initial and final state vectors during evolution. This is the result of our choice of a more constrained Hamiltonian than the one assumed in the general quantum brachistochrone problem. It is worth noting that such an evolution, being more complicated, is time optimal but with larger time than in the general case. This might be important for experiments, where a general Hamiltonian with all the allowed parameters is difficult to implement, but a constrained one, depending on the magnetic field, can be realized. However, for the preconstrained Hamiltonian not all final states are accessible. The present result does not contradict the general statement of the quantum brachistochrone problem, but gives additional insight into possible realization of the time-optimal passage.

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Recently Carlini *et al.* [1] generalized the classical brachistochrone problem for the quantum case. The quantum brachistochrone problem can be formulated in the following way: What is the optimal Hamiltonian, under a given set of constraints, such that the evolution from a given initial state $|\psi_i\rangle$ to a given final one $|\psi_f\rangle$ is achieved in the shortest time? Using the variational method, the authors solved this problem for some specific examples of constraints which lead to a fixed distance between the largest and smallest energy levels of the Hamiltonian. In [2] it was shown that the quantum brachistochrone problem could be solved more directly using symmetry properties of the quantum state space. That paper was based on the idea considered in [3], where an elementary derivation was provided for passage of time from the initial quantum state into another orthogonal state.

Later, the variational method was extended to allow finding the time-optimal realization of a target unitary operation, when the available Hamiltonians are subjected to certain constraints dictated by either experimental or theoretical conditions [4]. In [5] the authors considered the brachistochrone problem for quantum evolution of mixed states. Very recently, Bender *et al.* studied the brachistochrone problem for a *PT*-symmetric non-Hermitian two-dimensional matrix Hamiltonian [6] and showed that, among the non-Hermitian *PT*-symmetric Hamiltonians satisfying the same energy constraint, the time evolution between the two fixed states could be made arbitrarily small. Such an interesting phenomenon was observed also for dissipative systems described by a non-Hermitian Hamiltonian with eigenvalues having a negative imaginary part [7]. Some discussion on this subject can be found also in [8,9].

*amfry@ift.uni.wroc.pl
†tkachuk@ktf.franko.lviv.ua

The important statement of [1,2] (see also [6]) is that finding the minimal time of general evolution reduces to finding the optimal time evolution for the Hamiltonian acting on the two-dimensional subspace spanned by the initial and final state vectors $|\psi_i\rangle$ and $|\psi_f\rangle$. This means that the optimal evolution that realizes a quantum brachistochrone can be written as a linear combination of $|\psi_i\rangle$ and $|\psi_f\rangle$ with time-dependent coefficients. One of the aims of our paper is to examine this statement in detail within a three-dimensional quantum system.

We consider the brachistochrone problem for the case when the optimal Hamiltonian belongs to a preconstrained class of Hamiltonians, less general, with a smaller than allowed number of free parameters that can be used for the problem. Such a case is important from the physical point of view when an experimentalist has the possibility to change only a few parameters of the Hamiltonian but not all. As an example of such a scenario, we consider a three-level system, namely, spin 1 in an external magnetic field described by a Hamiltonian of the following form:

$$H = \hbar\omega(\mathbf{n}\cdot\mathbf{s}), \tag{1}$$

where $\mathbf{s}$ are dimensionless spin-1 operators, $\mathbf{n}$ is the direction of the magnetic field, and $\hbar\omega$ is proportional to the strength of the magnetic field. The eigenvalues of this Hamiltonian are $-\hbar\omega$, $\hbar\omega$, and 0. The difference between the largest and smallest eigenvalues is fixed as $\Delta E=\hbar\Delta\omega=2\hbar\omega$, which corresponds to the fixed absolute value of the magnetic field.

The Hamiltonian (1) contains only two free parameters, namely, two angles θ and ϕ that set the direction of the magnetic field,

$$n_x = \sin\theta\cos\phi, \quad n_y = \sin\theta\sin\phi, \quad n_z = \cos\theta. \tag{2}$$

Note that a general Hamiltonian in a three-dimensional space can be represented by a 3×3 Hermitian matrix, which con-

tains nine free parameters [eight if we consider the su(3) case]. We examine here the preconstrained class of Hamiltonians (1) with only two free parameters.

The brachistochrone problem in this restricted case is as follows: What is the optimal choice of the preconstrained Hamiltonian, namely, what is the optimal direction of the magnetic field $\mathbf{n}$ at the fixed ω, such that the evolution from a given initial state $|\psi_i\rangle$ to a given final one $|\psi_f\rangle$ is achieved in the shortest time? Obviously, with such a restriction of possible evolutions, not all general final states can be reached. This is the price for taking a narrower family of Hamiltonians. However, as we will show below, the shortest time achieved in optimal evolution in our case is the same as in the general setting, despite the fact that evolution in our case is more complicated (the system leaves the subspace spanned by the initial and final states when time is not the shortest one). Let us consider it in detail.

The vector of the state for spin 1 contains four parameters. We can write

$$|\psi\rangle = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} |a|e^{i\alpha_1} \\ |b|e^{i\alpha_2} \\ |c|e^{i\alpha_3} \end{pmatrix} = e^{i\alpha_1}\begin{pmatrix} |a| \\ |b|e^{i\alpha} \\ |c|e^{i\alpha'} \end{pmatrix}, \tag{3}$$

where the normalization condition is $|a|^2+|b|^2+|c|^2=1$. Hence four independent parameters, for instance $|a|$, $|b|$, α, and α', define the quantum state. Therefore, with only two parameters to change the Hamiltonian we cannot reach an arbitrary quantum state starting from a given initial one; in other words, the evolution defined by the Hamiltonian (1) cannot relate two arbitrary quantum states. In our restricted quantum brachistochrone problem, we consider only the states that can be connected by the implemented preconstrained evolution.

The evolution of the state vector can be realized as follows:

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi_i\rangle = e^{-i\omega(\mathbf{n}\cdot\mathbf{s})t}|\psi_i\rangle. \tag{4}$$

It is convenient to represent the unitary operator of evolution in the form

$$e^{-i\omega(\mathbf{n}\cdot\mathbf{s})t} = 1 - (\mathbf{n}\cdot\mathbf{s})^2 2\sin^2\frac{\omega t}{2} - i(\mathbf{n}\cdot\mathbf{s})\sin\omega t. \tag{5}$$

In order to prove this let us note that $\mathbf{n}\cdot\mathbf{s}$ is the operator of the projection of spin 1 on the direction $\mathbf{n}$ and it has three eigenvalues $-1,0,1$ with the corresponding eigenvectors $|-1\rangle,|0\rangle,|1\rangle$ which can play the role of the basis vectors. An arbitrary vector of state can be written as a linear combination of these vectors. It is enough to prove formula (5) only for the basis vectors, which are eigenvectors of $\mathbf{n}\cdot\mathbf{s}$ with eigenvalues $-1,0,1$. It is easy to verify that for λ, which takes only three values $-1,0,1$, we have

$$e^{\lambda x} = (1-\lambda)(1+\lambda) + \frac{1}{2}\lambda(\lambda+1)e^{x} + \frac{1}{2}\lambda(\lambda-1)e^{-x}. \tag{6}$$

Then, using (6) for the unitary operator of evolution, we just obtain (5).

Let us take the initial vector of state as the eigenvector of s_z with the eigenvalue -1,

$$|\psi_i\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{7}$$

and the accessible final state in the form given by (3). Then using (4) and representation (5) for the operator of evolution, and the matrix representation for a spin in which s_z is diagonal, we finally find

$$|\psi(t)\rangle = \begin{pmatrix} -e^{-i2\phi}\sin^2\theta\sin^2\dfrac{\omega t}{2} \\ \sqrt{2}e^{-i\phi}\cos\theta\sin\theta\sin^2\dfrac{\omega t}{2} - \dfrac{i}{\sqrt{2}}e^{-i\phi}\sin\theta\sin\omega t \\ 1-(1+\cos^2\theta)\sin^2\dfrac{\omega t}{2} + i\cos\theta\sin\omega t \end{pmatrix}. \tag{8}$$

The first component gives the necessary condition that $|\psi(t)\rangle$ reaches the final state

$$\sin^2\theta\sin^2\frac{\omega t}{2} = |a|. \tag{9}$$

From (8) it follows that the second component depends on the first one. Substituting $\sin^2\frac{\omega t}{2}$ from (9) into the second component of (8), we have

$$|b|^2 = 2|a|(1-|a|). \tag{10}$$

Then the normalization condition yields the relation

$$|c|^2 = 1-|a|^2-|b|^2 = (1-|a|)^2. \tag{11}$$

Thus we cannot reach an arbitrary state, but only those that have components satisfying conditions (10) and (11). In addition, note that the phases for the second and third components are not independent but related according to (8). If all necessary conditions are satisfied, then the time of evolution from the initial state to the allowed final one can be found from (9) as

$$t_f = \frac{4}{\Delta\omega}\arcsin\left(\frac{\sqrt{|a|}}{\sin\theta}\right), \tag{12}$$

where $\sin\theta > \sqrt{|a|}$, and $\hbar\Delta\omega = 2\hbar\omega$ is the distance between the largest and smallest energy levels.

It is interesting to note that this expression is very similar to the corresponding one for spin 1/2 [see, for instance, [6], Eq. (5)]. The difference is that (12) contains $\sqrt{|a|}$ instead of $|a|$ and is twice larger (in [6] a is denoted as b and $\Delta\omega$ is denoted as ω).

We obtain the minimal time that just corresponds to the quantum brachistochrone for $\theta=\pi/2$, when the magnetic field is perpendicular to the z axis,

$$t_{\min} = \frac{4}{\Delta\omega}\arcsin(\sqrt{|a|}). \tag{13}$$

As an explicit example let us consider the case $|a|=1$; then $|b|=|c|=0$. The vector of the final state in this case,

$$|\psi_f\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{14}$$

is the eigenvector of s_z with the eigenvalue 1; it is orthogonal to the initial one (7). We have a solution for time only when $\theta=\pi/2$. Thus the initial state (7) evolves to the final one (14) only when the magnetic field is perpendicular to the z axis. For the time of evolution we have $t_f=t_{\min}=2\pi/\Delta\omega$. This time is twice longer than the shortest possible time obtained in [1,2]. Note that the state vector describing the evolution in our case is not a superposition of the initial and final states. Therefore, it is not strange that the time of evolution is longer than the minimal possible one.

Let us consider the next example with the initial state

$$|\psi_i\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{15}$$

Now the evolution is given by the state vector

$$|\psi(t)\rangle = \begin{pmatrix} -\frac{1}{\sqrt{2}}e^{-i\phi}\left(2\cos\theta\sin\theta\sin^2\frac{\omega t}{2} + i\sin\theta\sin\omega t\right) \\ 1-2\sin^2\theta\sin^2\frac{\omega t}{2} \\ \frac{1}{\sqrt{2}}e^{i\phi}\left(2\cos\theta\sin\theta\sin^2\frac{\omega t}{2} - i\sin\theta\sin\omega t\right) \end{pmatrix} = \begin{pmatrix} -z^* \\ 1-2\gamma \\ z \end{pmatrix}, \tag{16}$$

where

$$z = \frac{1}{\sqrt{2}}e^{i\phi}\left(2\cos\theta\sin\theta\sin^2\frac{\omega t}{2} - i\sin\theta\sin\omega t\right) = |z|e^{i(\phi-\alpha)},$$

$$|z|^2 = 2\gamma(1-\gamma), \quad \gamma = \sin^2\theta\sin^2\frac{\omega t}{2}, \tag{17}$$

$$\tan\alpha = \frac{\cos(\omega t/2)}{\cos\theta\sin(\omega t/2)}.$$

Finally, the evolution of the state vector can be represented in the form

$$|\psi(t)\rangle = (1-2\gamma)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \sqrt{2\gamma(1-\gamma)}\begin{pmatrix} -e^{-i(\phi-\alpha)} \\ 0 \\ e^{i(\phi-\alpha)} \end{pmatrix}, \tag{18}$$

where γ and α are functions of time as given in (17). Let us consider the final state

$$|\psi_f\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \tag{19}$$

which is orthogonal to the initial one (15). In order to reach this state we put $\gamma(t_f)=1/2$. This condition gives us the time of evolution

$$t_f = \frac{4}{\Delta\omega}\arcsin\left(\frac{1}{\sqrt{2}\sin\theta}\right). \tag{20}$$

Then choosing additionally that $\phi=\alpha(t_f)$ we find that $|\psi(t_f)\rangle=|\psi_f\rangle$.

For $\theta=\pi/2$ we obtain the minimal time of evolution

$$t_{\min} = \frac{\pi}{\Delta\omega}. \tag{21}$$

It is interesting to note that this time is equal to the minimal possible time which can be obtained according to the statement [1,2] where the state vector of evolution for the minimal possible time belongs to the subspace spanned by the initial and final state vectors or, in other words, the vector of evolution for the minimal possible time is a superposition of the initial and final states. In our first example, the state vector during evolution does not stay in the subspace spanned by the initial and final state vectors. Therefore, we can conclude that, in order to achieve the minimal possible time during the evolution with preconstrained Hamiltonian, it is not necessary for the state vector to remain all the time in the subspace spanned by the initial and final state vectors. A preconstrained family of Hamiltonians can yield a more complicated evolution with the optimal time which is larger than the shortest time obtained for the general family of Hamiltonians. This fact can have a practical value for experiments. There is no contradiction with the results of [2,3]; the present result also shows that, to stay on the subspace spanned by the initial and final states, one should essentially use the full freedom in the general family of Hamiltonians for the system under consideration.

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