PAPER

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Excited-state quantum phase transitions in finite many-body systems

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Abstract

Quantum spectra of excited states of numerous collective many-body models show singularities related to stable and unstable stationary points of the corresponding classical dynamics. We show several examples of these singularities and discuss some of their consequences.

Keywords: quantum phase transitions, semiclassical state density, level dynamics

1. Introduction

Critical phenomena in many-body systems constitute a fascinating and quickly developing field of quantum physics [1]. The term quantum phase transition (QPT) generally covers various types of non-analyticities of the many-body ground state with respect to a non-thermal control parameter, like an external field intensity or an internal coupling strength [2–4]. These sharp singularities emerge in the system's infinite size limit, but distinct precursors of critical behavior are observed already in finite systems, such as atomic nuclei. In nuclear physics, the notion of QPT is invoked particularly in connection with transitions between spherical and deformed equilibrium shapes of nuclei with changing nucleon numbers [5, 6]. However, nuclear physics can serve also as a useful incubator of new QPT-related general ideas.

This latter function was proven a few years ago, when some models of nuclear collective dynamics became a hint for a generalization of the QPT concept to the excited domain [7–11]. The so-called excited-state quantum phase transitions (ESQPTs) represent singularities of quantal spectra of excited states in the direction of varying energy E (non-analyticities in the quantum density of states) as well as in the direction of varying control parameter λ (non-analyticities in an averaged 'flow' of the spectrum). These singularities turn out to be connected with stationary points of the corresponding classical Hamiltonians. The critical borderlines locating ESQPTs in the $\lambda \times E$ plane are usually (though not always) linked up with critical points of the ground-state QPTs, which justifies a unified treatment of both QPT and ESQPT phenomena. The ESQPT behavior has been identified in a wide variety of many-body, condensed-matter and quantum-optical models that exhibit also the ground-state QPT [12–27]. The excited-state singularities result in specific dynamical [20, 27] and thermodynamical [28, 29] effects, which constitute interesting new topics of theoretical and experimental research.

This paper summarizes basics of the ESQPT theory (in section 2), presents a simple pedagogical example (section 3), and outlines some more realistic cases (section 4). Note that a more comprehensive analysis of ESQPTs (although focused primarily on some restricted Hamiltonian forms) is presented in our recent articles [29, 30].

2. Outline of theory

Excited-state QPTs appear in *'finite systems of infinite size'*. This seemingly oxymoronic specification captures a large class of many-body models whose number of degrees of freedom f is finite and independent of a parameter $\aleph$ characterizing the overall size of the system. Even if the size parameter increases asymptotically, making the system 'infinite', the number of degrees of freedom remains constant, constituting the system's 'finiteness'. Such models typically describe some collective modes of many-body systems with a variable number of constituents, like collective motions of nuclei and molecules or coherent behavior of condensed-matter and quantum-optical systems.

The class of collective models with a limited number of degrees of freedom (or more precisely—models in which a few collective degrees of freedom decouple from an arbitrarily large complete set of all degrees of freedom) represents the stage on which the ESQPTs act. It turns out that the signatures of ESQPTs go weaker as f increases, so a small

value of f is essential. On the other hand, for really discontinuous (non-analytic) behavior one needs that the size parameter $\aleph$ becomes infinite since otherwise any discontinuity would be smoothened by quantum fluctuations.

The parameter $\aleph$ in general quantifies to which extent a given realization of the system under study is 'macroscopic'. It can be introduced as a characteristic area A of the two-dimensional classical phase associated with each degree of freedom measured in units of the elementary quantum action $\hbar$. Let us note that here we assume an approximately balanced distribution of action between individual degrees of freedom (some consequences of an imbalanced distribution are discussed in [30]). This means that for a system with f degrees of freedom a $2f$-dimensional volume of the phase space that is available to the system under some typical circumstances scales as $\Omega \sim (\aleph\hbar)^f$. Below we outline two archetypes of collective many-body systems and quantify their degree of freedom and size attributes.

(i) Systems composed of a fixed number of interacting bosons. The bosons usually represent some collective modes of a many-body system, as it is in the nuclear interacting boson model and in the molecular vibron model. Assume n types of bosons (part of these 'types' may represent spin components) and interactions conserving the total number of bosons N. This constitutes a realization of algebra $U(n)$ with generators of the form $b^\dagger_\mu b_\nu$ (products of creation and annihilation operators with $\mu, \nu = 1, ..., n$) in the Hilbert space corresponding to N bosons. Each boson type carries one degree of freedom, but one pair of canonical variables is eliminated due to the conservation of N. Hence $f = n - 1$. For a given value of N, the characteristic area of the phase space associated with each boson type is $A \sim N\hbar$ (one boson of the given type represents action $\sim\hbar$). Thus we have $\aleph \equiv N$: the size parameter in the interacting boson systems coincides with the total number of bosons.

(ii) Collective lattice systems. Consider a mesh of N spatial sites, each of them equipped with an n-state quantum system, e.g., an n-level atom or spin $j = \frac{n-1}{2}$. Quantum states of individual sites live in identical replicas of an n-dimensional Hilbert space, giving rise to an N-fold product of spaces carrying an elementary representation of algebra $U(n)_i$ associated with each site i. If the dynamics of the whole lattice is generated solely by the sums of individual $U(n)_i$ generators, the relevant algebra becomes the single $U(n)$ composed of these sums. Then the total wavefunction of the whole lattice can be assumed to be symmetric under the exchange of sites and the system can be interpreted as one of type (i). We again have $\aleph \equiv N$, which means that the size parameter in collective lattice systems coincides with the number of sites.

Let us note that proper thermodynamic properties of the above systems require that interactions between bosons or spin sites are reduced by suitable powers of the number N, namely by a factor $\sim N^{-k}$ at each term expressing the k-body interaction. This requirement follows from the unrestricted character of interactions which affect all bosons or act across the whole lattice. Without the scaling, the increase of N would lead to complete dominance of the highest-order interaction term in the total energy, which would divert the system's behavior on the way to the thermodynamic limit.

Considering a specific model of either type (i) or (ii), one can introduce collective f-dimensional coordinates and momenta, $\boldsymbol{q}$ and $\boldsymbol{p}$, and to express the collective Hamiltonian as $H^\lambda(\boldsymbol{p}, \boldsymbol{q})$. Here we denote explicitly the dependence on the control parameter λ. Let us stress that in general there is no restriction on the mutual coupling between various collective degrees of freedom (components of the vectors $\boldsymbol{q}$ and $\boldsymbol{p}$) in the Hamiltonian. A typical feature of algebraically formulated systems is that even the position- and momentum-dependent terms are interconnected in such an irritating way that kinetic and potential parts of the Hamiltonian cannot be easily disentangled. However, if we assume the standard dependence $H^\lambda = \frac{1}{2m}\boldsymbol{p}^2 + V^\lambda(\boldsymbol{q})$, where m is a mass parameter and V a certain potential, the expression for the size parameter $\aleph$ (based on the above phase-space criterion) is proportional to $\sqrt{m}\,\hbar^{-1}$ [29]. This hints at the fact that for the present type of systems, the thermodynamic limit $\aleph \to \infty$ is synonymous with the classical limit $\hbar \to 0$. A general proof of this universal feature can be deduced from the possibility to express the collective Hamiltonian in terms of $\frac{1}{N}$-scaled collective generators (cf section 3). As $N \to \infty$, the scaled generators yield vanishing commutators, which means that they effectively lose their quantum attributes.

The statement made in the last paragraph has crucial consequences. It implies that all quantum critical properties of finite collective systems must be rooted in the semiclassical limit of these systems. Since the defining property of an ESQPT is the density of quantum states as a function of energy, let us focus on this quantity first. Its semiclassical expression reads as:

$$\begin{aligned}\bar{\rho}^\lambda(E) &= \frac{1}{(2\pi\hbar)^f}\frac{\partial}{\partial E}\int\int \mathrm{d}^f\boldsymbol{q}\mathrm{d}^f\boldsymbol{p} \\ &\quad\times \Theta\left[E - H^\lambda_{\mathrm{clas}}(\boldsymbol{p}, \boldsymbol{q})\right] \\ &= \frac{1}{(2\pi\hbar)^f}\int_{H^\lambda_{\mathrm{clas}}(\boldsymbol{p},\boldsymbol{q})=E} \mathrm{d}^{2f-1}\boldsymbol{\sigma}^\lambda_E(\boldsymbol{q}, \boldsymbol{p})\ |\nabla H^\lambda_{\mathrm{clas}}(\boldsymbol{p}, \boldsymbol{q})|^{-1}, \quad (1)\end{aligned}$$

where Θ in the first line is the step function having the support in the phase-space domain with $H_{\mathrm{clas}} \leqslant E$, while the integration in the second line is performed over the $(2f-1)$-dimensional energy hypersurface $H_{\mathrm{clas}} = E$, with $\boldsymbol{\sigma}$ denoting local coordinates on this hypersurface. The inverse gradient of the classical Hamiltonian in the second integral, which results from the substitution formula for delta function $\delta(f(\boldsymbol{x})) = \delta(\boldsymbol{x} - \boldsymbol{x}_0)|\nabla f|^{-1}$, where f is a continuously differentiable function with $f(\boldsymbol{x}_0) = 0$, measures how quickly at the given place $(\boldsymbol{q}, \boldsymbol{p})$ the hypersurface with increasing energy moves away from the reference hypersurface with energy E. If $\nabla H_{\mathrm{clas}} = 0$ at any place of the hypersurface, that is, if the system at given energy has a stationary point, the subintegral function diverges and the hypersurface evolves singularly at this place. This will cause a kind of non-analyticity of the quasiclassical state density, which we interpret as an ESQPT.

We anticipate that non-analyticities in the density of states associated with various types of stationary points get weaker as the number of degrees of freedom increases. Let us consider, for example, a minimum of the classical Hamiltonian $H_{\rm clas}$ at energy E_0. Close to the minimum, the Hamiltonian is assumed to have a locally separable form, $H_{\rm clas} \sim E_0 + \sum_i [a_i \delta q_i^{k_i} + b_i \delta p_i^{l_i}]$, where a_i and b_i are some coefficients and k_i and l_i the powers associated with infinitesimal deviations δq_i and δp_i from the minimum along the respective directions in the phase space. For $E < E_0$ the minimum does not contribute to the phase-space volume represented by the first integral in equation (1). For $E \geqslant E_0$, however, the minimum produces an additional volume, which grows with $\delta E = E - E_0$ proportionally to δE^K, where $K = \sum_{i=1}^{f} (k_i^{-1} + l_i^{-1})$ (in the direction of ith coordinate or momentum, the phase space spreads as $\delta E^{1/k_i}$ or $\delta E^{1/l_i}$, respectively). For a purely quadratic minimum, for instance, we have $K = f$. The state density above energy E_0 acquires an additional term which grows as $\bar{\rho}_{\rm irreg} \propto \delta E^{K-1}$, yielding a discontinuous derivative of order $K - 1$ at $E = E_0$. If we allow for $f \to \infty$, the effect of the minimum with finite (k_i, l_i) becomes virtually undetectable. The presence of ESQPT signatures in low-order derivatives of the state density for increasing numbers of freedom degrees would require an increasing order ('flatness') of the stationary point.

Any kind of non-analyticity in the energy dependence of the semiclassical state density $\bar{\rho}^{\lambda}(E)$ induces a non-analyticity in flow properties of the spectrum ('level dynamics') with varying control parameter λ. Consider an average slope of the spectrum around energy E at parameter value λ. It is calculated as a weighted average $\bar{\phi}^{\lambda}(E) = \sum_i w_i(E) \frac{\rm d}{{\rm d}\lambda} E_i^{\lambda}$ of slopes of individual energy levels E_i^{λ}, where the normalized weight coefficients $w_i(E)$ decrease with $\Delta E = |E_i^{\lambda} - E|$ (these coefficients may follow the Gaussian distribution of width σ larger than the mean spacing of levels in the selected part of the spectrum). Note that the Hellmann–Feynman formula expresses the slopes through the corresponding eigenstate averages of the operator $\frac{\rm d}{{\rm d}\lambda} H^{\lambda}$. Assume that the quantity $\bar{\phi}^{\lambda}(E)$, which we call the flow rate, is evaluated as a function of λ and E, i.e., at various places of the spectrum. It can be shown [29] that the flow rate together with the state density satisfy the following relation:

$$\frac{\partial}{\partial \lambda} \bar{\rho}^{\lambda}(E) + \frac{\partial}{\partial E} \Big[\bar{\rho}^{\lambda}(E) \bar{\phi}^{\lambda}(E) \Big] = 0. \tag{2}$$

This is a direct analog of the ordinary continuity equation, with $\bar{\phi}$ replacing the velocity field. As a consequence, specific non-analyticities of the state density $\bar{\rho}^{\lambda}(E)$, as determined from equation (1), are in a quantitative relation with corresponding non-analyticities of the flow rate $\bar{\phi}^{\lambda}(E)$. The ESQPT concept attempts to unify both these non-analytic features of the spectrum. A locus of non-analyticity in the $\lambda \times E$ plane defines a critical borderline of the associated ESQPT.

3. Simple example

To illustrate the above-outlined general arguments, consider a collective lattice model with $n = 2$. It describes an array of N particles with spin $\frac{1}{2}$ (or, equivalently, an ensemble of two-level atoms) with a Hamiltonian per particle (in arbitrary energy units):

$$\begin{aligned} H^{\lambda,\chi} &= \frac{1-\lambda}{N}\left(\frac{N}{2} + J_z\right) - \frac{\lambda}{N^2}\left[2J_x + \chi\left(\frac{N}{2} + J_z\right)\right]^2 \\ &= E_0^{\lambda,\chi} - \frac{1}{N} \boldsymbol{B}^{\lambda,\chi} \cdot \boldsymbol{J} \\ &\quad - \frac{\lambda}{N^2}\Big[4J_x^2 + \chi^2 J_z^2 + 2\chi\big(J_x J_z + J_z J_x\big)\Big]. \end{aligned} \tag{3}$$

$\boldsymbol{J} \equiv (J_x, J_y, J_z)$ stand for collective spin operators, equal to the sum $\boldsymbol{J} = \sum_{i=1}^{N} \boldsymbol{j}^i$ of individual spins $\boldsymbol{j} \equiv (j_x^i, j_y^i, j_z^i)$. Note that all spins are taken dimensionless. If the collective spin operators are expressed through the individual ones, the Hamiltonian (3), multiplied by N, describes an array of spins in an external magnetic field $\boldsymbol{B}^{\lambda,\chi} = (2\lambda\chi,\, 0,\, \lambda\chi^2 + \lambda - 1)$ that interact with each other across the whole lattice via a two-body term given in the second line of equation (3). Note that $E_0^{\lambda,\chi} = \frac{1}{4}(2 - 2\lambda - \lambda\chi^2)$ is just an unimportant energy offset. The peculiar parametrization by a pair of control parameters $\lambda \in [0, 1]$ and $\chi \in (-\infty, +\infty)$ will be clarified in section 4, where it will be related to the nuclear interacting boson model.

To make an immediate contact with nuclei (many-fermion systems), we recall yet another familiar realization of the quasispin algebra $(J_x, J_y, J_z) \leftrightarrow (J_-, J_0, J_+)$ in terms of a pair of N-fold degenerate fermionic shells. The fermionic mapping reads as: $J_{\pm} = \sum_{i=1}^{N} a_{\pm i}^{\dagger} a_{\mp i}$ and $J_0 = \frac{1}{2}\sum_{i=1}^{N} (a_{+i}^{\dagger} a_{+i} - a_{-i}^{\dagger} a_{-i})$, where $a_{\pm i}^{\dagger}$ or $a_{\pm i}$ creates or annihilates a fermion in state i on the shell distinguished by the sign $\pm$. This is the Lipkin–Meshkov–Glick model [31], the most celebrated toy interacting fermion system.

Since $\boldsymbol{J}^2$ is conserved by the Hamiltonian (3), we consider in the following only the $(N+1)$-dimensional subspace with spin $J = \frac{N}{2}$ (the maximal possible value) in the full 2^N-dimensional Hilbert space of all spin sites. This subspace is fully symmetric under the exchange of spin sites. Moreover, any superposition in this subspace can be interpreted via the Bloch sphere as the spin pointing to a certain direction. In the fermionic representation, the spin conservation corresponds to the particle number conservation and the maximum-spin assumption is equivalent to considering N fermions in the system, i.e., to filling the two shells to a half of their full capacity.

The phase structure of our toy model is not difficult to anticipate. In case of vanishing interaction, $\lambda = 0$, the ground state has the spin parallel with the field $\boldsymbol{B}$. However, as the interaction strength increases, the quadratic spin terms included in the interaction Hamiltonian induce a gradual deviation of the ground state from the field direction. It turns out that in the limit $N \to \infty$, this process has an abrupt,

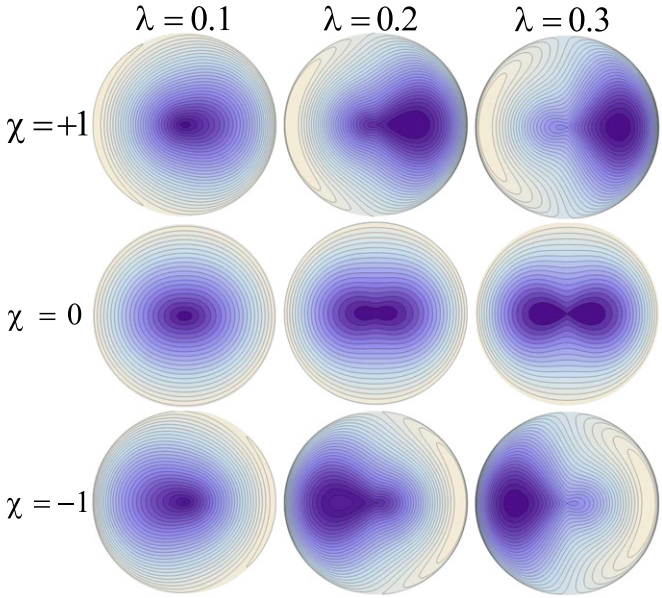

Figure 1. Contours of the classical Hamilton functions (4) of the spin model (3) with various choices of (λ, χ) parameters. Spherical angles (θ, ϕ) parameterizing the model phase space are mapped onto polar coordinates $r = 1 + \cos\theta$ and $\varphi = \phi$ on each disk.

phase-transitional character. The critical value of the coupling strength where the sudden spin change takes place is $\lambda_c = (5 + \chi^2)^{-1}$, the associated QPT being of the first order (changing the first derivative of the ground-state energy) for $\chi \neq 0$ and of the second order (changing the second derivative) for $\chi = 0$. On the other hand, if parameter χ is evolved between positive and negative values for a fixed $\lambda > \lambda_c$, the ground state undergoes a first-order QPT at $\chi_c = 0$. At this point, the ground-state spin flips between orientations with $\langle J_x \rangle > 0$ and $\langle J_x \rangle < 0$.

To demonstrate the above behavior, we parametrize the spin operators in the $J = \frac{N}{2}$ subspace by spherical angles (θ, ϕ), yielding $(J_x, J_y, J_z) = \frac{N}{2}(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Starting from commutation relations of the spin operators, it is easy to show that operators ϕ and $\cos\theta$ have the commutation relation of coordinate and momentum, respectively, the role of the Planck constant being played by $\sim\frac{2}{N}$. Hence in the limit $N \to \infty$ we can treat these operators as classical quantities. This leads to the following expression of the classical limit of Hamiltonian (3):

$$H_{\rm clas}^{\lambda,\chi} = E_0^{\lambda,\chi} + \frac{1-\lambda-\lambda\chi^2}{2}\cos\theta - \lambda\chi\sin\theta\cos\phi - \lambda\left[\sin^2\theta\cos^2\phi + \frac{\chi^2}{4}\cos^2\theta + \frac{\chi}{2}\sin 2\theta\cos\phi\right]. \quad (4)$$

The energy dependence given by this formula is shown in figure 1 for several combinations of control parameters. In each plot of this figure, the spherical phase space of the model is projected onto a planar disk through a transformation described in the caption.

Figure 1 not only shows the evolution of the global minimum of the Hamilton function (4), supporting the above-outlined critical properties of the ground state, but also exhibits a number of local stationary points that give rise to singularities in $N \to \infty$ spectra of excited states. These stationary points define the ESQPTs associated with our model. Figure 2 displays four parameter-dependent quantum spectra of Hamiltonian (3) that exhibit distinct finite-size precursors of these singular structures.

The spectra in figure 2 are taken along four paths in the $\lambda \times \chi$ parameter space and all correspond to the total spin $J = 25$. Apparent disturbances in the trajectories of individual levels (sequences of sharp avoided crossings) appear precisely at the energies of local stationary points of the classical Hamilton function, namely at the energy of a secondary minimum and saddle point (see figure 1). These energies vary with the respective control parameter and form some critical borderlines in the $\lambda \times E$ or $\chi \times E$ plane. While the local minimum generates a step-like change of both the state density and the flow rate across the critical borderline, the saddle point leads to a cusp-like behavior of the state density and flow rate. All these structures get sharper with increasing N.

We note that spectral singularities in the Lipkin–Meshkov–Glick model were first noticed in [12] for a restricted version of the model, and that they were further elaborated in [13, 14]. Here we newly present results on Hamiltonian (3) which enable us to directly compare typical ESQPT structures accompanying first- and second-order QPTs. The present results can be compared with related findings in different models, see [10] and [29].

4. More realistic examples

To give some more realistic examples of systems that exhibit ESQPT structures, let us start with the interacting boson model of nuclear physics [32] and various versions of the vibron model in molecular physics [33]. All these models are

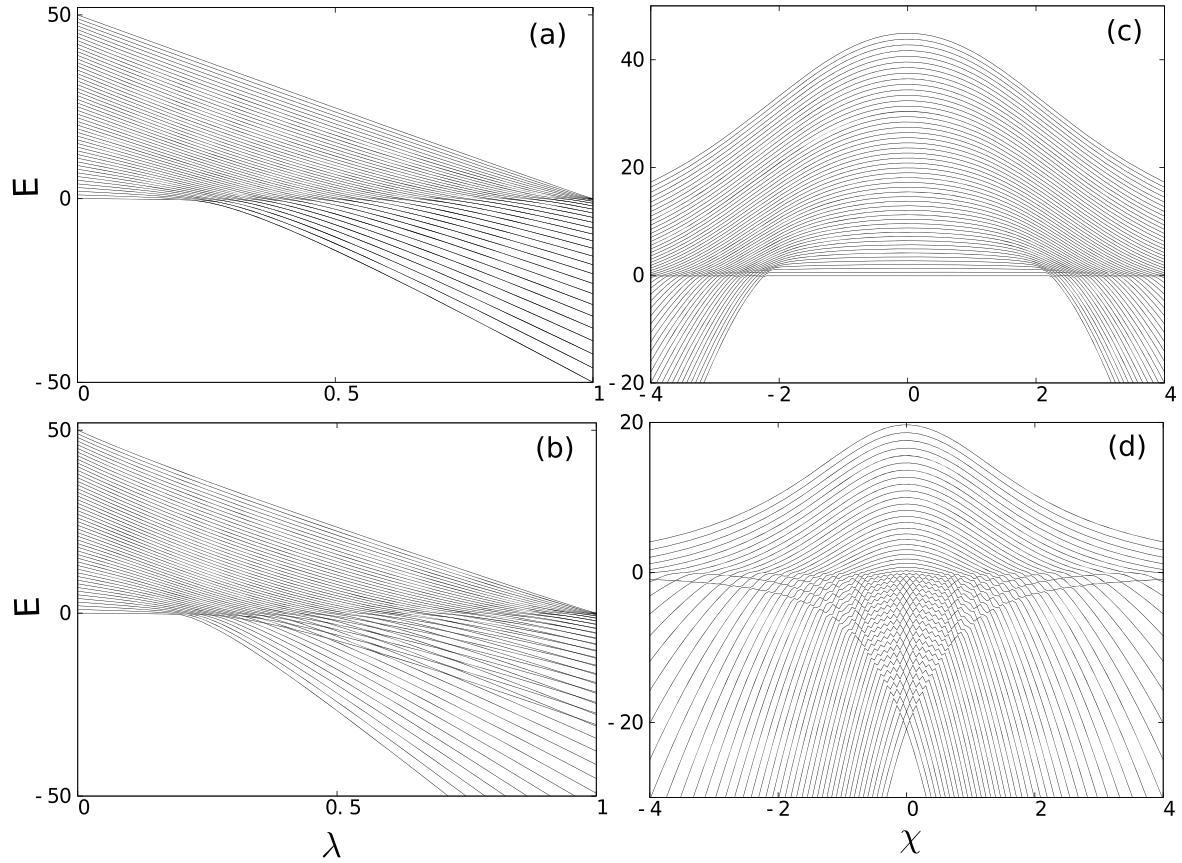

Figure 2. Evolutions of quantal spectra of the Hamiltonian (3) along various parameter paths for $N = 50$. Left panels: λ variable, χ fixed at $\chi = 0$ (a), $\chi = \pm 0.5$ (b). Right panels: χ variable, λ fixed at $\lambda = 0.6$ (c), $\lambda = 0.1$ (d). Singularities in all spectra are related to stationary points of the classical Hamiltonian (4), cf figure 1.

based on Hamiltonians describing an ensemble of bosons of two types. One of these bosons (called s) has a single component, the other one (called τ, p, or d, depending on the model) has $n-1$ components (for the above-mentioned cases $n-1 = 2$, 3, or 5, respectively). The number of degrees of freedom f coincides with $n-1$ [34].

In fact, the Lipkin–Meshkov–Glick model analyzed in section 3 can also be cast in the form of an interacting boson model with $n-1 = 1$. Indeed, using the Schwinger representation of the angular-momentum operators, $(J_-, J_0, J_+) \equiv (s^\dagger t, \frac{1}{2}(t^\dagger t - s^\dagger s), t^\dagger s)$, where $s^\dagger$ or s and $t^\dagger$ or t create or annihilate single-component bosons of two types (they can be taken as scalar and pseudoscalar, respectively), the Hamiltonian (3) becomes:

$$H^{\lambda,\chi} = \frac{1-\lambda}{N} t^\dagger t - \frac{\lambda}{N^2}\left(t^\dagger s + s^\dagger t + \chi t^\dagger t\right)^2. \qquad (5)$$

This is the form employed in nuclear and molecular boson models, provided that the t boson is replaced by the respective $(n-1)$-component boson (multiplications of boson operators are replaced by appropriate tensor couplings to ensure scalar character of the resulting Hamiltonian).

ESQPTs in molecular vibron models are theoretically investigated, e.g., in [9, 15, 23]. Experimental data demonstrating the presence of a singularity in highly excited spectra of simple molecules like H_2O or CO_2 are discussed, e.g., in [23, 35]. The ESQPT in these molecules is associated with a bent-to-linear shape transition that reorganizes characteristic structures in the spectrum. Note that this transition can be treated as a manifestation of a topological phenomenon called monodromy, see [7].

ESQPTs in the nuclear interacting boson model are partly analyzed in [7–9] and [21]. The results are known best for a specific bosonic Hamiltonian describing two-level pairing interaction. ESQPTs in analogous fermionic two-level pairing models are studied in [9, 19]. Similar structures as in the interacting boson model appear also in quadrupole geometric collective model [10]. An obvious obstacle to detect an ESQPT in atomic nuclei is the well-known aversion of these complex many-body objects to any kind of simplification. This is why nuclear spectra at high enough energies (where ESQPTs could perhaps be looked for) appear to be too complicated for any model based on a restricted number of degrees of freedom.

A more optimistic situation is met in quantum optics. The present-day quantum optical technologies start opening ways for experimental realizations of specific model Hamiltonians in the laboratory, making it gradually possible to 'engineer' the system of interest upon request. The goal of these efforts is to implement various quantum information techniques, including perhaps quantum computation.

For example, the Dicke and related models were proposed as a tool to schematically describe interaction of matter with electromagnetic field in a cavity [36, 37]. The Hamiltonian per atom can be expressed as:

$$H^{\lambda,\delta} = \frac{\omega_0}{N} b^\dagger b + \frac{\omega}{N} J_0 + \frac{\lambda}{N^{3/2}} \times \left[\left(b^\dagger J_- + b J_+ \right) + \delta \left(b^\dagger J_+ + b J_- \right) \right], \qquad (6)$$

where operators $b^\dagger$ and b create and annihilate bosons (photons) with energy ω_0, while the quasispin operators (J_-, J_0, J_+) describe a system of $N = 2J$ two-level atoms with the gap between levels ω. Each component of the atom-field system brings one degree of freedom, so the model has $f = 2$. Control parameters $\lambda \in [0, \infty)$ and $\delta \in [0, 1]$ adjusts the strength and form of the atom-field interaction. The model shows a well-known ground-state QPT (second-order) from the normal to superradiant phase at $\lambda_c = \omega_0\omega/(1 + \delta)$, where the average photon number $\langle b^\dagger b \rangle$ in the ground state changes from zero ($\lambda < \lambda_c$) to a nonzero value ($\lambda > \lambda_c$), and also specific ESQPTs connected with the stationary points obtained from the classical limit of equation (6) [20, 25, 26]. Although the critical behavior was for long considered as a mere artifact of the oversimplified form of the Hamiltonian, the ground-state QPT was recently detected experimentally using a Bose–Einstein condensate coupled to an optical cavity [38]. Explorations of the ESQPT may be considered as the next task for such experiments.

Let us note that the physics of Bose–Einstein condensation copes with the ESQPT concept also in other models and situations, see [16–18] and [22]. Another interesting application appeared recently in connection with the physics of graphene [24]. In this case, experimental spectrum of an idealized model of graphene based on a so-called superconducting microwave Dirac billiard was interpreted with the aid of a finite-size scaling typical for an ESQPT. Work on this subject is in rapid progress.

5. Conclusions

To summarize, we have seen that spectra of many-body systems with a finite number of collective degrees of freedom often exhibit distinct singularities—ESQPTs—which are connected with stationary points of the classical Hamiltonian. These singularities affect the density of states and the flow properties of the spectrum along certain critical borderlines in the plane of control parameter versus energy. They are of different types, depending on the type of stationary point and on the number of degrees of freedom f. In general, the larger is f, the weaker are the ESQPT signatures. Therefore, the key condition for the ESQPT effects to be distinguishable in realistic systems is an effective separation of a few collective degrees of freedom from the microscopic ones. We have outlined several models in which various ESQPTs occur and can be accessed experimentally.

We should stress that the investigation of ESQPTs is just at the beginning. It turns out that spectral singularities strongly affect the thermodynamical properties of the system (for instance, they cause discrepancies between canonical and microcanonical pictures) [28, 29] and the dynamical response of the system to external probes (like quantum quench [20] or periodic perturbation [27]). Although ESQPTs in collective spectra of atomic nuclei are probably too high in energy to be observable, one may ask whether some ESQPT-like dynamical effects associated with stationary points of energy landscapes could be identified in large-amplitude motions of nuclei, like fission.

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