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The chirality of exceptional points

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Abstract. Exceptional points are singularities of the spectrum and wave functions of a Hamiltonian which occur as functions of a complex interaction parameter. They are accessible in experiments with dissipative systems. We show that the wave function at an exceptional point is a specific superposition of two configurations. The phase relation between the configurations is equivalent to a chirality which should be detectable in an experiment.

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Rapid Note

The topological structure of an exceptional point (EP) has been explored in a recent experiment [1] for the first time. The experiment was designed to verify the theoretically predicted properties [2,3] of wave functions in the vicinity of an EP. In the present paper, we show that the wave function at the EP is a superposition that specifies a chirality. We argue that the chirality should be detectable in a suitable experiment.

We recapitulate the definition of an EP and summarise the results of the experiment that has verified its predicted topological structure. The formal result of the present note is derived for the case of a two-state system and generalised to an N-state system. We conclude proposing an experiment that would in principle detect the claimed chirality.

An EP is a branch point singularity of both, eigenvalue and eigenvector of a Hamiltonian, as functions of a complex interaction parameter λ. Suppose that the Hamiltonian is written as

$$H(\lambda) = H_0 + \lambda H_1. \tag{1}$$

Then the eigenvalues $E_k = E_k(\lambda)$ and the eigenvectors $|\psi_k\rangle = |\psi_k(\lambda)\rangle$ are functions of λ. Let H_0 and H_1 be real, symmetric matrices and λ a complex number. A coalescence of two eigenstates occurs at complex $\lambda = \lambda_\mathrm{c}$ when two eigenvalues (say $k = 1, 2$) coincide, *i.e.*

$$E_1(\lambda_c) = E_2(\lambda_c), \tag{2}$$

and the space of eigenvectors is only one-dimensional [4–6]. It is well-known that this cannot occur when H is Hermitian. In this case, equation (2) implies a degeneracy, *i.e.* a space of eigenvectors which is at least two-dimensional. Hence, an EP does not occur for real λ. In contrast, at complex values of λ, EP's do occur generically.

We emphasize that a coalescence should not be confused with a degeneracy. The latter gives rise to a diabolic point the properties of which have been predicted by Berry [7] and experimentally confirmed in reference [8]. The EP is a generalisation of a diabolic point (DP) in that the confluence of two complex conjugate EP at a real value of λ — *i.e.* the limiting case of a Hermitian Hamiltonian — yields a degeneracy. The DP has the conical structure of the energy surfaces described by Berry.

EPs are always found in the vicinity of a level repulsion: suppose that two levels show avoided level crossing when λ is varied along the real axis; then the analytic continuation into the complex λ-plane yields a complex conjugate pair of EPs. The occurrence of EPs is not restricted to repulsions of bound states, a recent paper deals with the repulsion of resonant states [9].

There are a number of phenomena, where the physical effect of an EP has been observed or discussed. Laser induced ionisation states of atoms [10] are a clear manifestation of an EP even though in [10], it has not been analysed as such. A recent theoretical paper [11] shows that, for a suitable choice of parameters associated with an EP, the only acoustic modes in an absorptive medium are circularly polarised waves with one specific orientation for a given EP. Similarly in optics, experimental observations in absorptive media [12] reveal the existence of handedness since the stable mode of light propagation is either a left or a right circularly polarised wave for appropriately chosen parameters. This has been interpreted in [13] in terms of EPs. A particular resonant behaviour of atom waves in crystals of light [14] has recently been interpreted [3] in terms of EPs. While absorption is essential in all cases, some situations clearly point to a chiral behaviour of an EP.

In the experiment [1], EPs have been investigated in a flat microwave cavity. The electric field distribution is

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analogous to the wave function and the eigenfrequencies are analogous to the eigenenergies of a quantum system. In fact, the flat cavity simulates the quantum mechanical motion of a particle in a billiard [15,16]. The eigenstates of the system are resonances with decay widths large enough to make quite a few EPs accessible. The parameter region surrounding one of the EPs has been explored.

The experiment [1] yielded three major results:

1. if a loop is performed in the λ-plane around the EP, the eigenenergies E_1 and E_2 are interchanged;
2. the wave functions $|\psi_1\rangle$ and $|\psi_2\rangle$ are interchanged by the loop and, in addition, one of them changes sign. In other words, a loop in the λ-plane transforms the pair $\{\psi_1, \psi_2\}$ into $\{-\psi_2, \psi_1\}$. Therefore the two possible directions of looping yield different phase behaviour. In fact, encircling the EP a second time in the same direction, we obtain $\{-\psi_1, -\psi_2\}$ while the next loop yields $\{\psi_2, -\psi_1\}$ and only the fourth loop restores the original pair. It follows that by going in the opposite direction, one finds after the first loop what is obtained after three loops in the former direction;
3. the eigenvalues E_1, E_2 have been studied as functions of λ for two paths that were not closed. One path was just above, the other one just below λ_c. The results were different. Let us call resonance energy the real part of an eigenvalue and resonance width its imaginary part. On one of the paths, the widths cross while the resonance energies avoid each other. On the other path, the resonance energies cross while the widths avoid each other.

These results are the consequence of the topological structure of Riemann sheets at a branch point. The experiment thus showed that this topology is a physical reality.

In the sequel, we carry the theoretical analysis of the EP one step further and focus upon the eigenfunction at the EP. Under item (2) above, a distinction is made between a right and a left turn. In part of the literature cited above, a handedness was observed [11,12] in connection with an EP. We show that a chiral structure is indeed an intrinsic property of every EP.

Since H is not self-adjoint for complex λ, the right hand eigenvectors $|\psi_k\rangle$ are different from the left hand eigenvectors $|\tilde{\psi}_k\rangle$. Both together form a biorthogonal basis, *i.e.* the completeness relation reads

$$\sum_k \frac{|\psi_k\rangle\langle\tilde{\psi}_k|}{\langle\tilde{\psi}_k|\psi_k\rangle} = 1 \tag{3}$$

for $\lambda \neq \lambda_c$. Recall the orthogonality relation

$$\langle\tilde{\psi}_k|\psi_{k'}\rangle = 0, \quad k \neq k'. \tag{4}$$

Due to the symmetric form of H, the left hand eigenvector

$$|\tilde{\psi}_k\rangle = |\psi_k^*\rangle \tag{5}$$

is merely the complex conjugate of its right hand partner. Hence, the (complex) components of the row vector $\langle\tilde{\psi}_k|$ coincide with the components of the column vector $|\psi_k\rangle$.

For $\lambda \neq \lambda_c$, one can define eigenvectors that are normalised in the biorthogonal sense. Indeed, the right hand eigenvectors

$$|\chi_k(\lambda)\rangle = |\psi_k\rangle / \sqrt{\langle\tilde{\psi}_k|\psi_k\rangle} \tag{6}$$

together with the left hand partners

$$\langle\tilde{\chi}_k(\lambda)| = \langle\tilde{\psi}_k| / \sqrt{\langle\tilde{\psi}_k|\psi_k\rangle} \tag{7}$$

form a biorthogonal system with the property

$$\langle\tilde{\chi}_k|\chi_{k'}\rangle = \delta_{kk'}, \quad \lambda \neq \lambda_c. \tag{8}$$

However, the χ_k diverge at the EP. It is not possible to define the eigenfunctions such that they are normalised for $\lambda \neq \lambda_c$ and also continuous at λ_c. This can be verified by explicitly working out a two-state system as in [2,3]. We assume that the $|\psi_k\rangle$ have been chosen continuous at λ_c.

From equation (4) follows that

$$\langle\tilde{\psi}_{\rm EP}|\psi_{\rm EP}\rangle = 0 \tag{9}$$

since the orthogonality (4) holds identically in λ and $|\psi_{\rm EP}\rangle$ is proportional to the limit of both eigenvectors, $|\psi_1\rangle$ and $|\psi_2\rangle$ for $\lambda \to \lambda_c$. As a consequence, the inverse of the biorthogonal norm $\langle\tilde{\psi}_k|\psi_k\rangle$ that appears in equation (3), does not exist at $\lambda = \lambda_c$.

In the case of a two-state system, equations (5, 9) entail

$$|\psi_{\rm EP}\rangle \propto \begin{pmatrix} \pm\mathrm{i} \\ 1 \end{pmatrix}. \tag{10}$$

Of course, this result can also be obtained from a representation of a symmetric 2×2 matrix in terms of its elements $H_{kk'}$. One requires that the eigenvalues coincide. If $H_{12} = 0$ one finds a 2-dimensional space of eigenvectors, *i.e.* a DP. If $H_{12} \neq 0$ one finds a 1-dimensional eigenvector space, *i.e.* an EP. The eigenvector has the form (10).

Equation (10) is true irrespective of a particular basis, since the vector (10) is invariant under all orthogonal transformations including complex ones. These are the transformations that conserve the symmetry of H which we are considering. Hence, in every basis with respect to which H is symmetric, $|\psi_{\rm EP}\rangle$ will have the form (10).

There is no orthogonal transformation that maps $\binom{\mathrm{i}}{1}$ onto $\binom{-\mathrm{i}}{1}$. Every $|\psi_{\rm EP}\rangle$ is therefore either $\sim \binom{\mathrm{i}}{1}$ or $\sim \binom{-\mathrm{i}}{1}$. This is the basic result of the present paper. It remains to be shown for an N-state system.

The eigenvector $|\psi_{\rm EP}\rangle$ is expanded according to

$$|\psi_{\rm EP}\rangle = \sum_k c_k(\lambda)|\chi_k(\lambda)\rangle, \tag{11}$$

where

$$c_k(\lambda) = \langle\tilde{\chi}_k|\psi_{\rm EP}\rangle. \tag{12}$$

For $\lambda \to \lambda_c$ the two coalescing eigenvectors $|\chi_1(\lambda)\rangle$ and $|\chi_2(\lambda)\rangle$ diverge. All other $N-2$ wave functions $|\chi_k\rangle$ remain regular. Therefore in the vicinity of the EP, only the terms with $k = 1, 2$ make substantial contributions to the expansion (11). In fact, the c_k vanish for $k \neq 1, 2$ and $\lambda \to \lambda_c$ as follows from the orthogonality (4) of different eigenvectors. For $k = 1, 2$ the c_k also vanish owing to equation (9) but this is exactly compensated by the diverging eigenfunctions $|\chi_1(\lambda)\rangle$ and $|\chi_2(\lambda)\rangle$.

Hence, the N-dimensional vector $|\psi_{\mathrm{EP}}\rangle$ becomes a superposition of the two N-dimensional vectors $|\chi_1(\lambda)\rangle$ and $|\chi_2(\lambda)\rangle$ in the limit of λ towards λ_c. From the result of the two-state model, we thus expect that

$$c_1/c_2 = +\mathrm{i} \quad \text{or} \quad c_1/c_2 = -\mathrm{i} \tag{13}$$

holds in the vicinity of λ_c.

A more explicit analytic consideration shows how this result comes about. We denote the components of $|\psi_{\mathrm{EP}}\rangle$ by $\{x_k\}, k = 1, \ldots, N$ and recall from equation (9) that $\sum_k x_k^2 = 0$. If λ is near to λ_c, the components of $|\psi_1(\lambda)\rangle$ can be chosen as $\{x_k + d_k\}$ with $d_k = a_k\sqrt{\lambda - \lambda_c} + O(\lambda - \lambda_c)$, $k = 1, \ldots, N$ and some constants a_k. The components of $\langle\tilde{\psi}_2(\lambda)|$ must therefore, to lowest order in the d_k, have the form $\{x_k - d_k\}$. To lowest order in the d_k we obtain

$$\langle\tilde{\psi}_1|\psi_1\rangle = 2\sum_k x_k d_k, \tag{14}$$

$$\langle\tilde{\psi}_2|\psi_2\rangle = -2\sum_k x_k d_k, \tag{15}$$

$$\langle\tilde{\psi}_1|\psi_{\mathrm{EP}}\rangle = \sum_k x_k d_k, \tag{16}$$

$$\langle\tilde{\psi}_2|\psi_{\mathrm{EP}}\rangle = -\sum_k x_k d_k \tag{17}$$

from which the statement of equation (13) immediately follows.

Each individual EP is associated with a particular chirality. This does not mean that all EPs of a given system have the same chirality. In a high dimensional system, one expects a random occurrence of the two possible chiralities of the EPs. Note that only those EPs are experimentally accessible that lead to negative imaginary parts of the eigenenergies E_1, E_2.

The prediction of the present paper — only one specific mode can be excited at or in the immediate vicinity of an EP — can be tested in principle by a microwave experiment. The nodal pattern of the two wave functions that coalesce at the EP must be explored as in reference [1]. The cavity must not be superconducting. There must rather be so much resistance in its walls that H becomes sufficiently non-Hermitian to have accessible EP's — again as in [1]. Both coalescing modes are separately excited through appropriately positioned antennae but with controllable phase difference between the antennae. Sweeping the phase shift should produce a maximal signal detected by a third antenna coupled to one of the modes, when the phase difference equals one of the values $\pm\pi/2$.

Of course, it would be interesting to see the experiment done on a true quantum system. Two coupled quantum dots [18] appear to be the closest quantum analogue to the set-up used in the Darmstadt experiment.

Let us conclude with a justification of the term "chirality" for the phase relation (10). If the waves, fed into the antennae of the microwave experiment, could be interpreted as two independent linear polarisations of a travelling microwave, the wave function (10) of the EP would, of course, represent a circularly polarised wave with a definite chirality [11,13]. If the antennae are simply coupled to two independent basis states of the system and if $|\psi_{\mathrm{EP}}\rangle$ is excited, the two basis configurations follow each other periodically with a time lag of a quarter of a period. The direction of positive time is defined by the experiment itself: in fact, the states are decaying as time goes by. The configurations $|\psi_1\rangle$ and $\psi_2\rangle$ are distinguishable by their nodal patterns. Hence, it is well defined which configuration leads and which one follows. In this sense, the configurations $|\chi_1\rangle, |\chi_2\rangle$ — linearly combined according to (10) — represent a motion on an abstract circle with a well defined handedness, *i.e.* a chirality.

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