Extended $\mathcal{PT}$- and $\mathcal{CPT}$-symmetric representations of fermionic algebras

O. Cherbal[1] and D. A. Trifonov[2]

[1]*Faculty of Physics, Theoretical Physics Laboratory, USTHB, Boîte Postale 32, El Alia, Algiers 16111, Algeria*

[2]*Institute of Nuclear Research, 72 Tzarigradsko chaussée, 1784 Sofia, Bulgaria*

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In a recent paper, Bender and Klevansky [Phys. Rev. A **84**, 024102 (2011)] considered $\mathcal{PT}$-symmetric matrix representations for fermionic operator algebras of the form $\xi^2 = \bar{\xi}^2 = 0$, $\xi\bar{\xi} + \bar{\xi}\xi = \varepsilon 1$, where $\bar{\xi}$ is the $\mathcal{PT}$ transform of ξ. They constructed such algebras for $\varepsilon = -1$ and established that it is not possible to construct a matrix representation for the standard fermionic algebra ($\varepsilon = 1$). Bender and Klevansky used the formalism developed by Jones-Smith and Mathur [Phys. Rev. A **82**, 042101 (2010)] which extends $\mathcal{PT}$-symmetric quantum mechanics to the case of odd time-reversal symmetry (fermionic case). By using the same formalism, we show that $\mathcal{PT}$-symmetric matrix representations exist for both standard ($\varepsilon = 1$) and abnormal ($\varepsilon = -1$) fermionic algebras if one takes $\bar{\xi}$ as adjoint ξ with respect to the $\mathcal{CPT}$ and $\mathcal{PT}$ inner products, respectively. This general result is illustrated for the example of a typical quaternionic four-level model by an explicit construction of the fermionic creation and annihilation operators which satisfy all the criteria of $\mathcal{PT}$ quantum mechanics for the odd time-reversal symmetry.

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I. INTRODUCTION

The quantum mechanics which deals with pseudo-Hermitian and $\mathcal{PT}$-symmetric Hamiltonians has attracted a great deal of interest over the last decade [1–3]. In this framework, Jones-Smith and Mathur [4] have developed a formalism in which they extend the construction of $\mathcal{PT}$-symmetric quantum mechanics to the case of odd time-reversal symmetry ($\mathcal{T}^2 = -1$), the case appropriate for fermions. Recently, Bender and Klevansky [5] have used the formalism developed in Ref. [4] in order to construct $\mathcal{PT}$-symmetric matrix representations for fermionic operator algebras of the form $\xi^2 = \bar{\xi}^2 = 0$, $\xi\bar{\xi} + \bar{\xi}\xi = \varepsilon 1$, where $\bar{\xi} = \xi^{\mathcal{PT}} = \mathcal{PT}\xi\mathcal{P}^{-1}\mathcal{T}^{-1}$, which is the $\mathcal{PT}$ reflection of ξ. In Ref. [5] it was shown that it is not possible to find matrix representations which correspond to $\varepsilon = 1$, but the matrix representation for such a fermionic algebra corresponds only to the value $\varepsilon = -1$.

On the other hand, fermionic algebra in quantum mechanics has been extended to the case of pseudo-Hermitian quantum mechanics by Mostafazadeh [6], showing that there exist two types of fermionic algebras which depend on whether the associated metric operator is definite or indefinite. In the first one, which corresponds to the definite metric operator η, the defining algebra is the pseudo-Hermitian generalization of the usual fermion algebra [6], namely, $\alpha^2 = \alpha^{\#2} = 0$, $\alpha\alpha^{\#} + \alpha^{\#}\alpha = 1$, where $\alpha^{\#} = \eta^{-1}\alpha^{\dagger}\eta$ is the pseudoadjoint of α with respect to the inner product $\langle\phi,\psi\rangle_\eta = \langle\phi,\eta\psi\rangle$, which means that $\alpha^{\#}$ and α satisfy $\langle\phi,\alpha\psi\rangle_\eta = \langle\alpha^{\#}\phi,\psi\rangle_\eta$ for all states ϕ and ψ. $\alpha^{\#}$ and α are, respectively, the creation and annihilation operators of what is called the pseudo-Hermitian fermion, or simply the phermion [6] . We remark that when $\eta = 1$, we have $\alpha^{\#} = \alpha^{\dagger}$, i.e., the phermion algebra reduces to the usual fermion algebra. The second type of fermionic algebra corresponds to the indefinite metric operator η, and the defining operator algebra is called an *abnormal* pseudo-Hermitian fermion [6], which satisfies $\alpha^2 = \alpha^{\#2} = 0$, $\alpha\alpha^{\#} + \alpha^{\#}\alpha = -1$. We would like to point out that the main guiding principle in this pseudo-Hermitian extension is to replace in the standard fermion algebra the adjoint operator $\alpha^{\dagger}$ by the pseudoadjoint one, $\alpha^{\#} = \eta^{-1}\alpha^{\dagger}\eta$.

In light of $\mathcal{PT}$ quantum mechanics with odd time-reversal symmetry introduced in Ref. [4] and the pseudo-Hermitian extension of the fermionic algebra achieved in Ref. [6], we introduce an alternative extension of fermionic algebra in the framework of $\mathcal{PT}$ quantum mechanics. The main guiding principle of our $\mathcal{PT}$-symmetric extension is to replace the adjoint operator $\alpha^{\dagger}$ in the standard fermion algebra (i) by the $\mathcal{PT}$ adjoint operator, denoted α_D in the indefinite metric case, and (ii) by the $\mathcal{CPT}$ adjoint operator, denoted $\alpha^{\ddagger}$ in the definite metric case.

This paper is organized as follows. In Sec. II, after reviewing the main definitions of the $\mathcal{PT}$ and $\mathcal{CPT}$ inner products and the $\mathcal{PT}$ and $\mathcal{CPT}$ adjoints in the $\mathcal{PT}$ quantum mechanics for the odd time-reversal symmetry, we introduce our $\mathcal{PT}$-symmetric extension of fermionic algebras. For illustration, we provide in Sec. III an explicit construction of the fermionic algebras in $\mathcal{PT}$ quantum mechanics in the typical quaternionic four-level model which satisfies all the conditions of $\mathcal{PT}$ quantum mechanics for the odd time-reversal symmetry [4]. We end with concluding remarks.

II. FERMIONIC ALGEBRAS IN $\mathcal{PT}$ QUANTUM MECHANICS

The $\mathcal{PT}$ inner product in the $\mathcal{PT}$ quantum mechanics for the odd time-reversal symmetry is defined by Ref. [4]

$$(\phi,\psi)_{\mathcal{PT}} = (\mathcal{PT}\phi)^T Z\psi = \phi^{\dagger} S\psi = \langle\phi,S\psi\rangle. \qquad (1)$$

Here $\langle\phi,\psi\rangle = \phi^{\dagger}\psi$ denotes the ordinary inner product. S and Z correspond to the linear operators used to define the parity and time-reversal operators, respectively [4]. The action of the time-reversal operator $\mathcal{T}$ on any state ψ is to multiply the complex conjugate of ψ by the matrix Z: $\mathcal{T}\psi = Z\psi^*$, where Z is the quaternion real diagonal matrix with all the diagonal terms equal to a 2×2 matrix e_2,

$$e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad (2)$$

and all the off-diagonal terms equal to zero [4]. The action of the parity operator P on any state ψ is to multiply ψ by the matrix S, which is real matrix given by

$$S = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{3}$$

where I is the 2×2 identity matrix.

The $\mathcal{PT}$ inner product is not positive definite. Then in view of Eq. (1), the $\mathcal{PT}$ inner product in the odd case is the pseudo-Hermitian inner product corresponding to the indefinite metric $\eta = S$. Moreover, the $\mathcal{PT}$ adjoint Y_D of an operator Y, defined through the $\mathcal{PT}$ inner product [4], takes the form of S-pseudoadjoint [7]:

$$Y_D = S^{-1} Y^\dagger S = S Y^\dagger S. \tag{4}$$

Since the metric is indefinite in this case, one deduces that the defining $\mathcal{PT}$-symmetric (fermionic) operator algebra is an *abnormal* fermion algebra [6],

$$Y^2 = Y_D^2 = 0, \quad Y Y_D + Y_D Y = -1. \tag{5}$$

In order to formulate a consistent $\mathcal{PT}$ quantum mechanics in the odd case, a positive-definite $\mathcal{CPT}$ inner product has been introduced in Ref. [4]. This inner product is used in lieu of the standard inner product. The $\mathcal{CPT}$ inner product is defined for the odd case as [4]

$$(\phi,\psi)_{\mathcal{CPT}} = (\mathcal{CPT}\phi)^T Z\psi = (CPZ\phi^*)^T Z\psi \tag{6}$$
$$= (KSZ\phi^*)^T Z\psi$$
$$= \phi^\dagger (KSZ)^T Z\psi = \phi^\dagger (Z^T S K^T) Z\psi, \tag{7}$$

where A^T is the transposed A. In view of the relations $Z^T = -Z$, $C\,PT = PT\,C$, and $SK^\dagger S = Z^T K^T Z = -ZK^T Z$ [4], the $\mathcal{CPT}$ inner product takes the compact form

$$(\phi,\psi)_{\mathcal{CPT}} = \phi^\dagger K^\dagger S \psi \equiv \langle \phi, K^\dagger S \psi \rangle. \tag{8}$$

The operator $K^\dagger S$ is positive definite, Hermitian $[(K^\dagger S)^\dagger = S^\dagger K = SK]$, and invertible. Therefore, denoting it as η_+, we can rewrite the odd time-reversal symmetry $\mathcal{CPT}$ scalar product (8) in the form of Mostafazadeh η_+ inner product [7], with $\eta_+ \equiv \mathcal{PC} = SK$,

$$(\phi,\psi)_{\mathcal{CPT}} = \phi^\dagger S K \psi = \langle \phi, \eta_+ \psi \rangle. \tag{9}$$

We have to note that our result (8) for the $\mathcal{CPT}$ inner product differs slightly from the result found in Ref. [4], where it has been given by $(\phi,\psi)_{\mathcal{CPT}} = \phi^\dagger K^T S \psi$, i.e., they found the transpose K^T in lieu of the adjoint $K^\dagger$ of K. As a consequence of our corrected result, we get that in the odd case the $\mathcal{CPT}$ adjoint $Y^\ddagger$ of an operator Y coincides with Mostafazadeh η_+ pseudoadjoint,

$$Y^\ddagger = (SK)^{-1} Y^\dagger S K = \eta_+^{-1} Y^\dagger \eta_+. \tag{10}$$

From this, and in accordance with Ref. [6], we deduce that (the fermionic operators) $Y^\ddagger$ and Y *satisfy the conventional fermionic algebra*, namely,

$$Y^2 = Y^{\ddagger 2} = 0, \quad Y Y^\ddagger + Y^\ddagger Y = 1. \tag{11}$$

III. ILLUSTRATION

In this section we illustrate the above general result by explicit matrix construction of the example of the simplest nontrivial quaternionic four-level model, which satisfies all the criteria of $\mathcal{PT}$ quantum mechanics for the fermionic odd time-reversal symmetry. The model is described by the following Hamiltonian [4]:

$$H = \begin{pmatrix} a & ib \\ ib^\dagger & -a \end{pmatrix}, \tag{12}$$

where $b = b_0\sigma_0 + ib_1\sigma_1 + ib_2\sigma_2 + ib_3\sigma_3$ is a real quaternion and $a = a_0\sigma_0$ is the real quaternion proportional to the identity, where σ_i $(i = 1,2,3)$ are the Pauli matrices. By setting $B_\pm = b_1 \pm ib_2$ and $C_\pm = b_3 \pm ib_0$, this Hamiltonian H can also be written as a four-level Hamiltonian as follows:

$$H = \begin{pmatrix} a_0 & 0 & -C_- & -B_- \\ 0 & a_0 & -B_+ & C_+ \\ C_+ & B_- & -a_0 & 0 \\ B_+ & -C_- & 0 & -a_0 \end{pmatrix}. \tag{13}$$

According to Ref. [4], the eigenvalues of H are $E = \pm\Omega$, with $\Omega = \sqrt{a_0^2 - |b|^2}$, where $|b|^2 = b_0^2 + b_1^2 + b_2^2 + b_3^2$, which is the magnitude of the quaternion b. These eigenvalues are twofold degenerate. As in Ref. [4], we deal with the real eigenvalues (i.e., $a_0^2 > |b|^2$).

We construct the $\mathcal{PT}$ doublets $(|\psi_1\rangle, -\mathcal{PT}|\psi_1\rangle)$ and $(|\psi_2\rangle, -\mathcal{PT}|\psi_2\rangle)$ associated with the negative and positive energies, respectively, as follows. For the negative energy $E = -\Omega$,

$$|\psi_1\rangle = \frac{1}{\sqrt{2\Omega}} \begin{pmatrix} \frac{\sqrt{a_0-\Omega}}{|b|}(iC_-) \\ \frac{\sqrt{a_0-\Omega}}{|b|}(iB_+) \\ i\sqrt{a_0+\Omega} \\ 0 \end{pmatrix}, \quad \mathcal{PT}|\psi_1\rangle = \frac{1}{\sqrt{2\Omega}} \begin{pmatrix} -\frac{\sqrt{a_0-\Omega}}{|b|}(iB_-) \\ \frac{\sqrt{a_0-\Omega}}{|b|}(iC_+) \\ 0 \\ -i\sqrt{a_0+\Omega} \end{pmatrix}. \tag{14}$$

For the positive energy $E = \Omega$,

$$|\psi_2\rangle = \frac{1}{\sqrt{2\Omega}} \begin{pmatrix} \frac{\sqrt{a_0+\Omega}}{|b|}(iC_-) \\ \frac{\sqrt{a_0+\Omega}}{|b|}(iB_+) \\ i\sqrt{a_0-\Omega} \\ 0 \end{pmatrix}, \quad \mathcal{PT}|\psi_2\rangle = \frac{1}{\sqrt{2\Omega}} \begin{pmatrix} -\frac{\sqrt{a_0+\Omega}}{|b|}(iB_-) \\ \frac{\sqrt{a_0+\Omega}}{|b|}(iC_+) \\ 0 \\ -i\sqrt{a_0-\Omega} \end{pmatrix}. \tag{15}$$

The action of $\mathcal{PT}$ on the eigenstates $|\psi_i\rangle$ $(i = 1,2)$ is given by

$$\mathcal{PT}\,|\psi_i\rangle = SZ\,|\psi_i\rangle^*, \tag{16}$$

where S and Z are given in the case of the four-level system by Ref. [4]

$$S = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad Z = \begin{pmatrix} e_2 & 0 \\ 0 & e_2 \end{pmatrix}, \tag{17}$$

where I_2 is 2×2 identity matrix and e_2 is the 2×2 matrix given in Eq. (2).

According to Ref. [4], we write the $\mathcal{PT}$ doublet $(|\psi_1\rangle, -\mathcal{PT}|\psi_1\rangle)$, which forms two column vectors as a single quaternion state $|\varphi_1\rangle$, and the $\mathcal{PT}$ doublet $(|\psi_2\rangle, -\mathcal{PT}|\psi_2\rangle)$, which forms two column vectors as a single quaternion state $|\varphi_2\rangle$. Explicitly, $|\varphi_1\rangle$ and $|\varphi_2\rangle$ are given by

$$\begin{aligned} |\varphi_1\rangle &= \frac{1}{\sqrt{2\Omega}} \begin{pmatrix} \frac{\sqrt{a_0-\Omega}}{|b|} b \\ i\sqrt{a_0+\Omega}\,\sigma_0 \end{pmatrix}, \\ |\varphi_2\rangle &= \frac{1}{\sqrt{2\Omega}} \begin{pmatrix} \frac{\sqrt{a_0+\Omega}}{|b|} b \\ i\sqrt{a_0-\Omega}\,\sigma_0 \end{pmatrix}. \end{aligned} \tag{18}$$

The quaternion columns $|\varphi_1\rangle$ and $|\varphi_2\rangle$ are also eigenstates of H associated with the negative and positive energies, respectively. It is useful to mention that the forms of the $\mathcal{PT}$ doublets $|\varphi_1\rangle$ and $|\varphi_2\rangle$ are in agreement with the general form of a state given in Ref. [4] which is invariant under $\mathcal{PT}$. In other words, the upper terms of $|\varphi_1\rangle$ and $|\varphi_2\rangle$ are real quaternions, and the lower terms are real quaternions multiplied by i.

The eigenstates ($\mathcal{PT}$ doublets) $|\chi_1\rangle$ and $|\chi_2\rangle$ associated with $H^\dagger$ are obtained from the action of S on $|\varphi_1\rangle$ and $|\varphi_2\rangle$, respectively, and are given by

$$\begin{aligned} |\chi_1\rangle &= \frac{1}{\sqrt{2\Omega}} \begin{pmatrix} \frac{\sqrt{a_0-\Omega}}{|b|} b \\ -i\sqrt{a_0+\Omega}\,\sigma_0 \end{pmatrix}, \\ |\chi_2\rangle &= \frac{1}{\sqrt{2\Omega}} \begin{pmatrix} \frac{\sqrt{a_0+\Omega}}{|b|} b \\ -i\sqrt{a_0-\Omega}\,\sigma_0 \end{pmatrix}. \end{aligned} \tag{19}$$

These states satisfies the relations

$$(\varphi_1,\varphi_1)_{\mathcal{PT}} = \langle \varphi_1, S\varphi_1\rangle = \langle \varphi_1,\chi_1\rangle = -1, \tag{20}$$

$$(\varphi_2,\varphi_2)_{\mathcal{PT}} = \langle \varphi_2, S\varphi_2\rangle = \langle \varphi_2,\chi_2\rangle = 1, \tag{21}$$

$$(\varphi_n,\varphi_m)_{\mathcal{PT}} = 0, (n \neq m). \tag{22}$$

Thus, $|\varphi_1\rangle$ has a negative $\mathcal{PT}$ norm, while $|\varphi_2\rangle$ has a positive $\mathcal{PT}$ norm. We note that H is pseudo-Hermitian with indefinite metric S:

$$H^\dagger = SHS^{-1}, \tag{23}$$

where S is given in Eq. (17).

We introduce the annihilation operator Y associated with the Hamiltonian H given in Eq. (12) by

$$Y = |\varphi_1\rangle\langle\chi_2| = \frac{1}{2\Omega} \begin{pmatrix} |b|\,\sigma_0 & \frac{i(a_0-\Omega)}{|b|} b \\ \frac{i(a_0+\Omega)}{|b|} b^\dagger & -|b|\,\sigma_0 \end{pmatrix}. \tag{24}$$

The operator Y can be written in four dimensions as

$$Y = \frac{1}{2\Omega} \begin{pmatrix} |b| & 0 & -\frac{(a_0-\Omega)}{|b|} C_- & -\frac{(a_0-\Omega)}{|b|} B_- \\ 0 & |b| & -\frac{(a_0-\Omega)}{|b|} B_+ & \frac{(a_0-\Omega)}{|b|} C_+ \\ \frac{(a_0+\Omega)}{|b|} C_+ & \frac{(a_0+\Omega)}{|b|} B_- & -|b| & 0 \\ \frac{(a_0+\Omega)}{|b|} B_+ & -\frac{(a_0+\Omega)}{|b|} C_- & 0 & -|b| \end{pmatrix}. \tag{25}$$

We remark that Y has a vanishing trace and determinant. Its adjoint operator reads

$$Y^\dagger = \frac{1}{2\Omega} \begin{pmatrix} |b|\,\sigma_0 & -\frac{i(a_0+\Omega)}{|b|} b \\ -\frac{i(a_0-\Omega)}{|b|} b^\dagger & -|b|\,\sigma_0 \end{pmatrix}, \tag{26}$$

and its $\mathcal{PT}$ adjoint Y_D (which is equal to the S-pseudoadjoint) is given by

$$Y_D = SY^\dagger S \tag{27}$$

and takes the form

$$Y_D = \frac{1}{2\Omega} \begin{pmatrix} |b|\,\sigma_0 & \frac{i(a_0+\Omega)}{|b|} b \\ \frac{i(a_0-\Omega)}{|b|} b^\dagger & -|b|\,\sigma_0 \end{pmatrix}. \tag{28}$$

After calculation, we find that Y_D and Y satisfy the abnormal fermionic algebra, namely,

$$Y^2 = Y_D^2 = 0, \quad YY_D + Y_D Y = -\mathbf{1}. \tag{29}$$

Y and Y_D act on states $|\varphi_i\rangle$ as follows:

$$Y\,|\varphi_1\rangle = 0, \qquad Y\,|\varphi_2\rangle = |\varphi_1\rangle\ , \tag{30}$$

$$Y_D\,|\varphi_2\rangle = 0, \qquad Y_D|\varphi_1\rangle = -|\varphi_2\rangle\ . \tag{31}$$

We embark now on the construction of the fermionic algebra in terms of operator Y and its $\mathcal{CPT}$ adjoint. As we have mentioned previously, the $\mathcal{CPT}$ adjoint $Y^\ddagger$ of an operator Y with respect to the $\mathcal{CPT}$ inner product is nothing but the η_+ pseudoadjoint with $\eta_+ = SK \equiv \mathcal{PC}$. Moreover, H is also pseudo-Hermitian with positive-definite metric operator η_+:

$$H^\dagger = \eta_+ H \eta_+^{-1}, \tag{32}$$

where η_+ and η_+^{-1} are given explicitly by the matrices

$$\begin{aligned} \eta_+ &= \frac{1}{\Omega} \begin{pmatrix} a & ib \\ -ib^\dagger & a \end{pmatrix}, \\ \eta_+^{-1} &= |\varphi_1\rangle\langle\varphi_1| + |\varphi_2\rangle\langle\varphi_2| = \frac{1}{\Omega} \begin{pmatrix} a & -ib \\ ib^\dagger & a \end{pmatrix}. \end{aligned} \tag{33}$$

Here $|\varphi_1\rangle$ and $|\varphi_2\rangle$ are the eigenstates ($\mathcal{PT}$ doublets) of H given in Eq. (18). They satisfy the biorthonormal relations

$$\langle\varphi_n|\lambda_m\rangle = \delta_{nm}, \quad \sum_n |\lambda_n\rangle\langle\varphi_n| = \sum_n |\varphi_n\rangle\langle\lambda_n| = \mathbf{1}, \tag{34}$$

where $|\lambda_1\rangle$ and $|\lambda_2\rangle$ are the eigenstates of $H^\dagger$. They are obtained through the action of η_+ on eigenstates $|\varphi_1\rangle$ and $|\varphi_2\rangle$, respectively, and read

$$\begin{aligned} |\lambda_1\rangle &= \frac{1}{\sqrt{2\Omega}}\begin{pmatrix} \frac{\sqrt{a_0-\Omega}}{|b|}(ib) \\ \sqrt{a_0+\Omega}\sigma_0 \end{pmatrix} = -|\chi_1\rangle, \\ |\lambda_2\rangle &= \frac{1}{\sqrt{2\Omega}}\begin{pmatrix} -\frac{\sqrt{a_0+\Omega}}{|b|}(ib) \\ -\sqrt{a_0-\Omega}\sigma_0 \end{pmatrix} = |\chi_2\rangle. \end{aligned} \tag{35}$$

Indeed, we find that the expression of η_+ given in Eq. (33) is equal to $SK \equiv \mathcal{P}C$, where S is given in Eq. (17) and K is the normalized Hamiltonian matrix:

$$K = \frac{H}{\Omega} = \frac{1}{\Omega}\begin{pmatrix} a & ib \\ ib^\dagger & -a \end{pmatrix}. \tag{36}$$

We introduce now the annihilation operator Y' associated with the Hamiltonian $H^\dagger$ as

$$Y' = |\varphi_1\rangle\langle\lambda_2|. \tag{37}$$

In view of Eq. (35) we see that Y' coincides with Y, Eq. (24). Next we have to construct the $\mathcal{CPT}$ adjoint $Y^\ddagger$ of Y. By taking into consideration Eqs. (10) and (33), we find that $Y^\ddagger$ is given explicitly by

$$Y^\ddagger = \frac{1}{2\Omega}\begin{pmatrix} -|b|\,\sigma_0 & -\frac{i(a_0+\Omega)}{|b|}b \\ -\frac{i(a_0-\Omega)}{|b|}b^\dagger & |b|\,\sigma_0 \end{pmatrix}. \tag{38}$$

After calculation, we find that $Y^\ddagger$ and Y satisfy the conventional fermionic algebra, namely,

$$Y^2 = Y^{\ddagger 2} = 0, \quad YY^\ddagger + Y^\ddagger Y = \mathbf{1}. \tag{39}$$

The action of Y and $Y^\ddagger$ on states $|\varphi_i\rangle$ is the same as that of the lowering and raising operators:

$$Y|\varphi_1\rangle = 0, \qquad Y|\varphi_2\rangle = |\varphi_1\rangle, \tag{40}$$

$$Y^\ddagger|\varphi_2\rangle = 0, \qquad Y^\ddagger|\varphi_1\rangle = |\varphi_2\rangle. \tag{41}$$

Y is the lowering operator, annihilating the lowest eigenstates $|\varphi_1\rangle$, and $Y^\ddagger$ is the raising operator, mapping $|\varphi_1\rangle$ onto the upper eigenstates $|\varphi_2\rangle$.

IV. CONCLUDING REMARKS

In this article, we have achieved the fermionic algebra in the framework of the odd time-reversal $\mathcal{PT}$ quantum mechanics. We have shown that, like the case of fermionic algebras in the pseudo-Hermitian quantum mechanics [6], there exist two types of fermionic algebras which depend on whether the associated metric operator is definite or indefinite. In the definite metric case, the defining algebra is of the form $Y^2 = Y^{\ddagger 2} = 0$, $YY^\ddagger + Y^\ddagger Y = 1$, where $Y^\ddagger = KSY^\dagger SK$ is the $\mathcal{CPT}$ adjoint of Y, with S and K being the matrix representations of the P and C operators, respectively. In the indefinite metric operator case, the defining operator algebra is of the form $Y^2 = Y_D^2 = 0$, $YY_D + Y_DY = -1$, where $Y_D = SY^\dagger S$ is the $\mathcal{PT}$ adjoint of Y.

Moreover, we have shown that, as in the $\mathcal{PT}$ quantum mechanics with even time-reversal symmetry, the $\mathcal{CPT}$ inner product in the $\mathcal{PT}$ quantum mechanics with odd time-reversal symmetry is nothing but the positive-definite η_+ pseudo-Hermitian inner product with $\eta_+ = \mathcal{P}C$. We have illustrated our general results in the interesting model of a quaternionic four-level system which meets all the conditions of $\mathcal{PT}$ quantum mechanics for the odd time-reversal symmetry [4], providing five parameter families of 4×4 matrices that realize the $\mathcal{PT}$-symmetric representations of both standard and abnormal fermionic algebras.

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