Faster than Hermitian Quantum Mechanics

Carl M. Bender,[1,2] Dorje C. Brody,[2] Hugh F. Jones,[3] and Bernhard K. Meister[4]

[1]*Physics Department, Washington University, St. Louis, Missouri 63130, USA*
[2]*Department of Mathematics, Imperial College, London SW7 2BZ, United Kingdom*
[3]*Blackett Laboratory, Imperial College, London SW7 2BZ, United Kingdom*
[4]*Department of Physics, Renmin University of China, Beijing 100872, China*

(Received 5 September 2006; published 24 January 2007)

Given an initial quantum state $|\psi_I\rangle$ and a final quantum state $|\psi_F\rangle$, there exist Hamiltonians H under which $|\psi_I\rangle$ evolves into $|\psi_F\rangle$. Consider the following quantum brachistochrone problem: subject to the constraint that the difference between the largest and smallest eigenvalues of H is held fixed, which H achieves this transformation in the least time τ? For Hermitian Hamiltonians τ has a nonzero lower bound. However, among non-Hermitian $\mathcal{PT}$-symmetric Hamiltonians satisfying the same energy constraint, τ can be made arbitrarily small without violating the time-energy uncertainty principle. This is because for such Hamiltonians the path from $|\psi_I\rangle$ to $|\psi_F\rangle$ can be made short. The mechanism described here is similar to that in general relativity in which the distance between two space-time points can be made small if they are connected by a wormhole. This result may have applications in quantum computing.

DOI: 10.1103/PhysRevLett.98.040403 PACS numbers: 03.65.Xp, 03.65.Ca, 03.67.Lx, 11.30.Er

Suppose that one wishes to transform unitarily a state $|\psi_I\rangle$ in a Hilbert space to a different state $|\psi_F\rangle$ by means of a Hamiltonian H. In Hermitian quantum mechanics, such a transformation requires a nonzero amount of time, provided that the difference between the largest and the smallest eigenvalues of H is held fixed. However, if we extend quantum mechanics into the complex domain while keeping the energy eigenvalues real, then under the same energy constraint it is possible to achieve such a transformation in an *arbitrarily short time*. In this Letter we demonstrate this by means of simple examples.

The Letter is organized as follows: we first review why in Hermitian quantum mechanics there is an unavoidable lower bound τ on the time required to transform one state into another. In particular, we consider the minimum time required to flip unitarily a spin-up state into a spin-down state. We then summarize briefly how Hermitian quantum mechanics can be extended into the complex domain while retaining the reality of the energy eigenvalues, the unitarity of time evolution, and the probabilistic interpretation. In this complex framework we show how a spin-up state can be transformed arbitrarily quickly to a spin-down state by a simple non-Hermitian Hamiltonian. Then we discuss the transformation between pairs of states by more general complex non-Hermitian Hamiltonians. We make some comments regarding possible experimental consequences of these ideas.

In Hermitian quantum mechanics how does one achieve the transformation $|\psi_I\rangle \rightarrow |\psi_F\rangle = e^{-iHt/\hbar}|\psi_I\rangle$ in the shortest time $t = \tau$? Since τ is the minimum of all possible evolution times t, the Hamiltonian associated with τ is the "quantum brachistochrone" [1]. Finding the optimal evolution time requires only the solution to a much simpler problem, namely, finding the optimal evolution time for the 2×2 matrix Hamiltonians acting in the two-dimensional subspace spanned by $|\psi_I\rangle$ and $|\psi_F\rangle$ [2].

To solve the Hermitian version of the two-dimensional quantum brachistochrone problem one can choose the basis so that the initial and final states are given by

$$|\psi_I\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\psi_F\rangle = \begin{pmatrix} a \\ b \end{pmatrix}, \tag{1}$$

with $|a|^2 + |b|^2 = 1$. The most general 2×2 Hermitian Hamiltonian has the form

$$H = \begin{pmatrix} s & re^{-i\theta} \\ re^{i\theta} & u \end{pmatrix}, \tag{2}$$

where the four parameters r, s, u, and θ are real. The eigenvalue constraint $E_+ - E_- = \omega$ reads

$$\omega^2 = (s-u)^2 + 4r^2. \tag{3}$$

The Hamiltonian H in (2) can be expressed in terms of the Pauli matrices as $H = \frac{1}{2}(s+u)\mathbf{1} + \frac{1}{2}\omega\boldsymbol{\sigma}\cdot\mathbf{n}$, where $\mathbf{n} = \frac{2}{\omega}(r\cos\theta, r\sin\theta, \frac{s-u}{2})$ is a unit vector. Using the matrix identity $\exp(i\phi\boldsymbol{\sigma}\cdot\mathbf{n}) = \cos\phi\mathbf{1} + i\sin\phi\boldsymbol{\sigma}\cdot\mathbf{n}$, the relation $|\psi_F\rangle = e^{-iH\tau/\hbar}|\psi_I\rangle$ takes the form

$$\begin{pmatrix} a \\ b \end{pmatrix} = e^{-i(s+u)t/(2\hbar)} \begin{pmatrix} \cos(\frac{\omega t}{2\hbar}) - i\frac{s-u}{\omega}\sin(\frac{\omega t}{2\hbar}) \\ -i\frac{2r}{\omega}e^{i\theta}\sin(\frac{\omega t}{2\hbar}) \end{pmatrix}. \tag{4}$$

From the second component of (4) we obtain $|b| = \frac{2r}{\omega} \times \sin(\frac{\omega t}{2\hbar})$, which gives the time required to transform the initial state: $t = \frac{2\hbar}{\omega}\arcsin(\frac{\omega|b|}{2r})$. We optimize this relation over all $r > 0$, keeping in mind that (3) gives a maximum value of $\frac{1}{2}\omega$ for r, achieved when $s = u$. The optimal time is thus

$$\tau = \frac{2\hbar}{\omega} \arcsin|b|. \tag{5}$$

Note that if $a = 0$ and $b = 1$ we have $\tau = \pi\hbar/\omega$ for the smallest time required to transform $\binom{1}{0}$ to the orthogonal state $\binom{0}{1}$. This value of τ is called the *passage time* [3].

For general a and b, at the optimal time τ we have $a = e^{-i\tau s/\hbar}\sqrt{1-|b|^2}$ and $b = -ie^{-i\tau s/\hbar}|b|e^{i\theta}$, which satisfies the condition $|a|^2 + |b|^2 = 1$ that the norm of the state does not change under unitary time evolution. The parameters s and θ are determined by the phases of a and b. Writing $a = |a|e^{i\arg(a)}$ and $b = |b|e^{i\arg(b)}$, we find that the optimal Hamiltonian is

$$H = \begin{pmatrix} \frac{\omega \arg(a)}{2\arcsin|b|} & \frac{\omega}{4}e^{-i[\arg(b)-\arg(a)-\pi/2]} \\ \frac{\omega}{4}e^{i[\arg(b)-\arg(a)-\pi/2]} & \frac{\omega \arg(a)}{2\arcsin|b|} \end{pmatrix}. \tag{6}$$

Since the overall phase of $|\psi_F\rangle$ is not physically relevant, the quantity $\arg(a)$, for example, may be chosen arbitrarily, and without loss of generality we may assume that it is 0. We are free to choose $\arg(a)$ because there is no absolute energy in quantum mechanics; one can add a constant to the eigenvalues of the Hamiltonian without altering the physics. Equivalently, this means that the value of τ cannot depend on the trace $s + u$ of H.

Interpreting the result for τ in (5) requires care because while this equation resembles the time-energy uncertainty principle, it is merely the statement that rate $\times$ time = distance. The constraint (3) on H is equivalent to placing a bound on the standard deviation ΔH of the Hamiltonian, where ΔH in a normalized state $|\psi\rangle$ is given by $(\Delta H)^2 = \langle\psi|H^2|\psi\rangle - \langle\psi|H|\psi\rangle^2$. The maximum value of ΔH is $\omega/2$. According to the Anandan-Aharonov relation [4], the speed of evolution of a quantum state is given by ΔH. The distance between the initial state $|\psi_I\rangle$ and the final state $|\psi_F\rangle$ is $\delta = 2\arccos(|\langle\psi_F|\psi_I\rangle|)$. Thus, the shortest time τ to achieve the evolution from $|\psi_I\rangle$ to $|\psi_F\rangle = e^{-iH\tau/\hbar}|\psi_I\rangle$ is bounded below because the speed is bounded above while the distance is held fixed. The Hamiltonian H that realizes the shortest time evolution can be understood as follows: the standard deviation ΔH of the Hamiltonian in (2) is r. Since ΔH is bounded by $\omega/2$, to maximize the speed of evolution (and minimize the time of evolution) we choose $r = \omega/2$.

The objective of this Letter is to perform the same optimization for complex non-Hermitian Hamiltonians having $\mathcal{PT}$ symmetry. There are infinitely many $\mathcal{PT}$-symmetric complex non-Hermitian Hamiltonians whose eigenvalues are real and bounded below. Here, $\mathcal{P}$ is the parity reflection operator and $\mathcal{T}$ is the time reversal operator. A one-parameter family of such Hamiltonians that has been investigated intensively [5,6] is given by $H = p^2 + x^2(ix)^\epsilon$, where $\epsilon > 0$. Although this Hamiltonian is not Hermitian in the usual Dirac sense, where Hermitian adjoint consists of complex conjugation and matrix transposition, H defines a unitary theory of quantum mechanics [7,8]. This is because H is self-adjoint with respect to a new inner product that is different from the Dirac inner product of conventional quantum mechanics.

This new inner product is expressed in terms of a linear operator $\mathcal{C}$ that satisfies three equations [9]:

$$\mathcal{C}^2 = 1, \qquad [\mathcal{C}, H] = 0, \quad \text{and} \quad [\mathcal{C}, \mathcal{PT}] = 0. \tag{7}$$

For any given H we can, in principle, calculate $\mathcal{C}$ by solving the three equations in (7). We then define an inner product in terms of $\mathcal{CPT}$ conjugation. Thus, in a $\mathcal{PT}$-symmetric quantum theory the inner product is specified dynamically in terms of the Hamiltonian. Furthermore, the time-evolution operator $e^{-iHt/\hbar}$ is unitary (norm preserving) because H commutes with $\mathcal{CPT}$.

It has been shown that for any $\mathcal{PT}$-symmetric Hamiltonian having real eigenvalues there exists an equivalent Hermitian Hamiltonian [10]. The argument goes as follows: the Hermitian operator $\mathcal{CP}$ is positive and can thus be written as $\mathcal{CP} = e^{Q}$. By means of the similarity transformation $\tilde{H} = e^{-Q/2}He^{Q/2}$ one can construct the corresponding Hermitian Hamiltonian $\tilde{H}$, which is equivalent to H in the sense that it has the same eigenvalues. The states in the $\mathcal{PT}$-symmetric theory are mapped by the operator $e^{-Q/2}$ to corresponding states in the Hermitian theory. But, since this operator does not keep the states in the same Hilbert space, relative properties of states can be changed. For example, the overlap distance between two states does not remain constant in the original Hilbert space. In this Letter we exploit this property of the transformation to circumvent the Hermitian limit on τ.

We now show how to solve the $\mathcal{PT}$-symmetric brachistochrone problem for a simple non-Hermitian two-dimensional matrix Hamiltonian of the form

$$H = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix}. \tag{8}$$

(This Hamiltonian was examined in detail in Ref. [7].) To show that H in (8) is $\mathcal{PT}$ symmetric, we let $\mathcal{T}$ be the operation of complex conjugation and $\mathcal{P}$ be given by

$$\mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{9}$$

The eigenvalues $E_\pm = r\cos\theta \pm \sqrt{s^2 - r^2\sin^2\theta}$ of H in (8) are real provided that $s^2 > r^2\sin^2\theta$. This inequality defines the region of unbroken $\mathcal{PT}$ symmetry.

The unnormalized eigenstates of H are

$$|E_+\rangle = \begin{pmatrix} e^{i\alpha/2} \\ e^{-i\alpha/2} \end{pmatrix}, \qquad |E_-\rangle = \begin{pmatrix} ie^{-i\alpha/2} \\ -ie^{i\alpha/2} \end{pmatrix}, \tag{10}$$

where the real parameter α is defined by $\sin\alpha = (r/s) \times \sin\theta$. The operator $\mathcal{C}$ satisfying the conditions in (7) is given by

$$\mathcal{C} = \frac{1}{\cos\alpha}\begin{pmatrix} i\sin\alpha & 1 \\ 1 & -i\sin\alpha \end{pmatrix}. \tag{11}$$

By using (9) and (11) we calculate that the $\mathcal{CPT}$ norms of both eigenstates in (10) are $\sqrt{2\cos\alpha}$.

Following the procedure used for Hermitian Hamiltonians, we rewrite H in (8) in the form $H = (r\cos\theta)\mathbf{1} + \frac{1}{2}\omega\boldsymbol{\sigma}\cdot\mathbf{n}$, where $\mathbf{n} = \frac{2}{\omega}(s, 0, ir\sin\theta)$ is a unit vector and the squared difference between the energy eigenvalues is

$$\omega^2 = 4s^2 - 4r^2\sin^2\theta. \tag{12}$$

The positivity of ω^2 is ensured by the condition of unbroken $\mathcal{PT}$ symmetry. This equation emphasizes the key difference between Hermitian and $\mathcal{PT}$-symmetric Hamiltonians: the corresponding equation (3) for a Hermitian matrix Hamiltonian has a *sum* of squares, while this equation for ω^2 has a *difference* of squares. Thus, Hermitian Hamiltonians exhibit elliptic behavior, which leads to a nonzero lower bound for the optimal time τ. The hyperbolic nature of (12) allows τ to approach zero because, as we will see, the matrix elements of the $\mathcal{PT}$-symmetric Hamiltonian can be made large without violating the energy constraint $E_+ - E_- = \omega$.

The $\mathcal{PT}$-symmetric analog of the evolution Eq. (4) is given by

$$e^{-iHt/\hbar}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{e^{-itr\cos\theta/\hbar}}{\cos\alpha}\begin{pmatrix} \cos(\frac{\omega t}{2\hbar} - \alpha) \\ -i\sin(\frac{\omega t}{2\hbar}) \end{pmatrix}. \tag{13}$$

We apply this result to the pair of vectors examined in the Hermitian case: $|\psi_I\rangle = \binom{1}{0}$ and $|\psi_F\rangle = \binom{0}{1}$. (Note that these vectors are not orthogonal with respect to the $\mathcal{CPT}$ inner product.) From (13) we see that the evolution time to reach $|\psi_F\rangle$ from $|\psi_I\rangle$ is $t = (2\alpha + \pi)\hbar/\omega$. Optimizing this result over allowable values for α, we see that as α approaches $-\frac{1}{2}\pi$ the optimal time τ tends to zero.

Note that in the limit $\alpha \to -\frac{1}{2}\pi$ we get $\cos\alpha \to 0$. However, in terms of the variable α the energy constraint (12) becomes $\omega^2 = 4s^2\cos^2\alpha$. Since ω is held fixed, in order to have α approach $-\frac{1}{2}\pi$ we must require that $s \gg 1$. It then follows from the relation $\sin\alpha = (r/s)\sin\theta$ that $|r| \sim |s|$, so we must also require that $r \gg 1$. Evidently, in order to make $\tau \ll 1$ the matrix elements of the $\mathcal{PT}$-symmetric Hamiltonian (8) must be large.

The result demonstrated here does not violate the uncertainty principle. Indeed, a Hermitian Hamiltonian and a non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian both share the properties that (i) the passage time is given by $\pi\hbar/\omega$, and (ii) $\Delta H \leq \omega/2$. The key difference is that a pair of states such as $\binom{1}{0}$ and $\binom{0}{1}$ are orthogonal in a Hermitian theory but have the separation $\delta = \pi - 2|\alpha|$ in the $\mathcal{PT}$-symmetric theory. This is because the Hilbert-space metric of the $\mathcal{PT}$-symmetric quantum theory depends on the Hamiltonian. As a consequence, it is possible to set the parameter α to create a wormholelike effect in the Hilbert space [11].

A *gedanken* experiment to realize this effect in a laboratory might work as follows: we use a Stern-Gerlach filter to create a beam of spin-up electrons. The beam then passes through a "black box" containing a device governed by a $\mathcal{PT}$-symmetric Hamiltonian that flips the spins unitarily in a very short time. The outgoing beam then enters a second Stern-Gerlach device that verifies that the electrons are now in spin-down states. In effect, the black-box device is applying a magnetic field in the complex direction $(s, 0, ir\sin\theta)$. If the field strength is sufficiently strong, then spins can be flipped unitarily in virtually no time because the complex path joining these two states is arbitrary short without violating the energy constraint. The arbitrarily short alternative complex pathway from an up state to a down state, as illustrated by this thought experiment, is reminiscent of the short alternative distance between two widely separated space-time points as measured through a wormhole in general relativity [12].

The $\mathcal{PT}$-symmetric Hamiltonian (8) used in the foregoing illustrative example contains only three arbitrary real parameters, which are not sufficient to allow the initial state $|\psi_I\rangle = \binom{1}{0}$ to evolve into any final state $|\psi_F\rangle$. Indeed, it follows from (13) that $\binom{1}{0}$ can only evolve into $\binom{a}{b}$ if the relative phase of a and b is $\pm\frac{1}{2}\pi$. Therefore, we introduce the more general four-real-parameter $\mathcal{PT}$-symmetric Hamiltonian

$$H = \begin{pmatrix} x + (z + iy) & \frac{z}{\tan\gamma} - iy\tan\gamma \\ \frac{z}{\tan\gamma} - iy\tan\gamma & x - (z + iy) \end{pmatrix}, \tag{14}$$

which is associated with a more general definition of parity reflection $\mathcal{P}$ than that used in (9):

$$\mathcal{P} = \begin{pmatrix} \sin\gamma & \cos\gamma \\ \cos\gamma & -\sin\gamma \end{pmatrix}. \tag{15}$$

We retain the same definition for $\mathcal{T}$, namely, that $\mathcal{T}$ performs complex conjugation.

As before, we express H in the form $H = x\mathbf{1} + \frac{\omega}{2}\boldsymbol{\sigma}\cdot\mathbf{n}$, where in this case $\mathbf{n} = \frac{2}{\omega}(z/\tan\gamma - iy\tan\gamma, 0, z + iy)$. The operator $\mathcal{C} = \boldsymbol{\sigma}\cdot\mathbf{n}$ is given by

$$\mathcal{C} = \frac{2}{\omega}\begin{pmatrix} z + iy & \frac{z}{\tan\gamma} - iy\tan\gamma \\ \frac{z}{\tan\gamma} - iy\tan\gamma & -z - iy \end{pmatrix}, \tag{16}$$

which, along with (15), allows us to define the inner product with respect to which the Hamiltonian (14) becomes self-adjoint. The energy constraint $E_+ - E_- = \omega$ again takes a hyperbolic form:

$$\omega^2 = 4z^2\csc^2\gamma - 4y^2\sec^2\gamma. \tag{17}$$

For this more general Hamiltonian, the initial state $\binom{1}{0}$ evolves as follows:

$$e^{-iHt/\hbar}\begin{pmatrix}1\\0\end{pmatrix}=e^{-ixt/\hbar}\begin{pmatrix}\cos(\frac{\omega t}{2\hbar})+\frac{2y}{\omega}\sin(\frac{\omega t}{2\hbar})-i\frac{2z}{\omega}\sin(\frac{\omega t}{2\hbar})\\-\frac{2y}{\omega}\tan\gamma\sin(\frac{\omega t}{2\hbar})-i\frac{2z}{\omega\tan\gamma}\sin(\frac{\omega t}{2\hbar})\end{pmatrix}. \tag{18}$$

The time evolution preserves the $\mathcal{CPT}$ norm of the initial state, which is $(\frac{2z}{\omega\sin\gamma})^{1/2}$. We therefore choose the general form of the final state to be

$$|\psi_F\rangle=\sqrt{\frac{2z}{\omega\sin\gamma}}\begin{pmatrix}ue^{iA}\\ve^{i(A+\xi)}\end{pmatrix}, \tag{19}$$

where u, v, A, and ξ are real parameters.

We now introduce dimensionless variables $X=2x/\omega$, $Y=2y/\omega$, $Z=2z/\omega\sin\gamma$, and $T=\omega t/2\hbar$, as well as the shifted variable $B=A+xt/\hbar$. By identifying the right sides of (18) and (19), we obtain

$$\begin{aligned}\cos T+Y\sin T&=\sqrt{Z}u\cos B,\\ Y\tan\gamma\sin T&=-\sqrt{Z}v\cos(B+\xi),\\ \sqrt{Z}\sin\gamma\sin T&=-u\sin B,\\ \sqrt{Z}\cos\gamma\sin T&=-v\sin(B+\xi).\end{aligned} \tag{20}$$

The energy constraint in (17) takes the hyperbolic form

$$1=Z^2-Y^2\sec^2\gamma. \tag{21}$$

The condition that the norm of the initial vector be preserved under time evolution imposes the requirement that $u^2+v^2+2uvY\sin\xi/Z\cos\gamma=1$, which can be derived from the five equations in (20) and (21).

The generic problem is now to pick a final vector, that is, to choose the parameters A, u, v, and ξ in (19), and then to solve (20) and (21) to determine the parameters X, Y, Z, and γ for which $|\psi_I\rangle$ reaches $|\psi_F\rangle$ under the Hamiltonian in (14). We then must find the smallest value of T for which the transformation is realized.

To illustrate the procedure, we consider the example $u=v$ and solve the five simultaneous equations in the form of Laurent series valid for small T. The result is

$$\begin{aligned}X&=(\tfrac{1}{4}\pi-\tfrac{1}{2}\xi-A)T^{-1}+O(T),\\ Y&=-uT^{-1}\cos(\tfrac{1}{4}\pi-\tfrac{1}{2}\xi)+O(T),\\ Z&=-uT^{-1}+O(T),\\ \gamma&=\tfrac{1}{4}\pi-\tfrac{1}{2}\xi+O(T^2).\end{aligned} \tag{22}$$

Note that the parameter T may be taken arbitrarily small and thus the initial vector evolves into the final vector in an optimal time that is arbitrarily small. Of course, the matrix elements of the Hamiltonian become large in this limit, but this is possible because the energy constraint in (17) is hyperbolic in character.

We conclude by remarking that the results established here provide the possibility of performing experiments that definitively distinguish between Hermitian and $\mathcal{PT}$-symmetric Hamiltonians. If practical implementation of complex $\mathcal{PT}$-symmetric Hamiltonians were proved feasible, then the identification of the optimal unitary transformation would be particularly important in the design and implementation of fast quantum communication and computation algorithms (cf. [13]).

Of course, the wormholelike effect we have discussed here can only be realized if we can switch rapidly between Hermitian and $\mathcal{PT}$-symmetric Hamiltonians by means of similarity transformations. It is conceivable that there is a sort of quantum protection mechanism that places a lower bound on the time required to switch Hilbert spaces. If so, this would limit the applicability of a Hilbert-space wormhole to improve quantum algorithms.

We thank D. W. Hook for useful discussions. C. M. B. thanks the US Department of Energy and D. C. B. thanks The Royal Society for support.

[1] A. Carlini, A. Hosoya, T. Koike, and Y. Okudaira, Phys. Rev. Lett. **96**, 060503 (2006).
[2] D. C. Brody and D. W. Hook, J. Phys. A **39**, L167 (2006).
[3] D. C. Brody, J. Phys. A **36**, 5587 (2003).
[4] J. Anandan and Y. Aharonov, Phys. Rev. Lett. **65**, 1697 (1990).
[5] C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998); C. M. Bender, S. Boettcher, and P. N. Meisinger, J. Math. Phys. (N.Y.) **40**, 2201 (1999).
[6] P. Dorey, C. Dunning, and R. Tateo, J. Phys. A **34**, L391 (2001); **34**, 5679 (2001).
[7] C. M. Bender, D. C. Brody, and H. F. Jones, Phys. Rev. Lett. **89**, 270401 (2002); Am. J. Phys. **71**, 1095 (2003).
[8] A. Mostafazadeh, J. Math. Phys. (N.Y.) **43**, 3944 (2002).
[9] C. M. Bender, D. C. Brody, and H. F. Jones, Phys. Rev. Lett. **93**, 251601 (2004).
[10] A. Mostafazadeh, J. Math. Phys. (N.Y.) **43**, 205 (2002); J. Phys. A **36**, 7081 (2003).
[11] A somewhat similar observation regarding the difference between unitary and antiunitary transformations was made by E. P. Wigner, J. Math. Phys. (N.Y.) **1**, 414 (1960).
[12] M. S. Morris, K. S. Thorne, and U. Yurtsever, Phys. Rev. Lett. **61**, 1446 (1988); M. S. Morris and K. S. Thorne, Am. J. Phys. **56**, 395 (1988).
[13] V. Giovannetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004); M. A. Nielsen, M. R. Dowling, M. Gu, and A. C. Doherty, Science **311**, 1133 (2006); U. Boscain and P. Mason, J. Math. Phys. (N.Y.) **47**, 062101 (2006).