

\mathcal{PT} -Symmetry Breaking and Laser-Absorber Modes in Optical Scattering SystemsY. D. Chong,^{*} Li Ge,[†] and A. Douglas Stone*Department of Applied Physics, Yale University, New Haven, Connecticut 06520, USA*

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Using a scattering matrix formalism, we derive the general scattering properties of optical structures that are symmetric under a combination of parity and time reversal (\mathcal{PT}). We demonstrate the existence of a transition between \mathcal{PT} -symmetric scattering eigenstates, which are norm preserving, and symmetry-broken pairs of eigenstates exhibiting net amplification and loss. The system proposed by Longhi [Phys. Rev. A **82**, 031801 (2010).], which can act simultaneously as a laser and coherent perfect absorber, occurs at discrete points in the broken-symmetry phase, when a pole and zero of the S matrix coincide.

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Schrödinger equations that violate time-reversal symmetry due to a non-Hermitian potential, but retain combined \mathcal{PT} (parity-time) symmetry, have been extensively studied since the work of Bender *et al.* [1,2], who showed that such systems can exhibit real energy eigenvalues, suggesting a possible generalization of quantum mechanics. Moreover, \mathcal{PT} -symmetric systems can display a spontaneous breaking of \mathcal{PT} symmetry, at which the reality of the eigenvalues is lost [1,3]. Although \mathcal{PT} -symmetric quantum mechanics remains speculative as a fundamental theory, the idea has been fruitfully extended to wave optics [3–6]. The classical electrodynamics of a medium with loss or gain breaks \mathcal{T} symmetry in the mathematical sense, although the underlying quantum electrodynamics is of course \mathcal{T} symmetric. \mathcal{PT} symmetry is maintained in optical systems by means of balanced gain and loss regions that transform into one another under parity; thus the combined \mathcal{PT} operation, which also interchanges loss and gain, leaves the system invariant.

We show in this Letter that the *scattering* behavior of a general \mathcal{PT} -symmetric system can exhibit one or multiple spontaneous symmetry-breaking transitions. This result applies to arbitrarily complex \mathcal{PT} -symmetric scattering geometries, whereas earlier works on optical \mathcal{PT} symmetry breaking were inherently restricted to waveguide (paraxial) geometries [3–6] for which resonances in the propagation direction play no role. In addition, we elucidate the properties of certain singular solutions occurring in such systems, recently studied for special cases by several authors [7–9], where a pole and a zero of the scattering matrix (S matrix) coincide at a specific real frequency. A real-frequency pole corresponds to the threshold for laser action, while a real-frequency zero implies the reverse process to lasing, in which a particular incoming mode is perfectly absorbed. A device exhibiting the latter phenomenon, which does not require \mathcal{PT} symmetry, has been termed a “coherent perfect absorber” (CPA) [10]. A \mathcal{PT} -symmetric scatterer, at these singular points, can function simultaneously as a CPA and a laser at threshold, as noted by Longhi [9]. The present work

establishes the CPA-laser points as special solutions in a wider “phase” of \mathcal{PT} -broken scattering eigenstates. We identify signatures of both the \mathcal{PT} -breaking transition and the CPA-lasing points, for several exemplary and experimentally feasible geometries.

S-matrix properties.—Consider an optical cavity coupled to a discrete set of scattering channels, denoted $\mu = 1, 2, \dots$. Incoming fields enter via the input channels, interact in the cavity, and exit via the output channels. For simplicity, we focus on the scalar wave equation, which directly describes one-dimensional (1D) and two-dimensional (2D) systems. A steady-state scattering solution for the electric field, $E(\vec{r})$, obeys

$$[\nabla^2 + n^2(\vec{r})(\omega^2/c^2)]E(\vec{r}) = 0. \quad (1)$$

For amplifying or dissipative media, $n(\vec{r})$ is complex. The frequency ω is real for physical processes but can be usefully continued to the complex plane. Henceforth, we set $c = 1$. Outside the cavity, E has the form

$$E(\vec{r}) = \sum_{\mu} [\psi_{\mu} u_{\mu}^{\text{in}}(\vec{r}, \omega) + \varphi_{\mu} u_{\mu}^{\text{out}}(\vec{r}, \omega)]. \quad (2)$$

Here $u_{\mu}^{\text{in}}(\vec{r}, \omega)$ and $u_{\mu}^{\text{out}}(\vec{r}, \omega)$ denote the input and output channel modes, whose exact forms depend on the scattering geometry (e.g., plane waves, spherical or cylindrical waves, or waveguide modes). The complex input and output amplitudes, ψ_{μ} and φ_{μ} , are related by the S matrix:

$$\sum_{\nu} S_{\mu\nu}(\omega) \psi_{\nu} = \varphi_{\mu}. \quad (3)$$

For all values of ω , $S(\omega)$ is a (complex) symmetric matrix [11]; if $n(\vec{r})$ and ω are real, it is also unitary.

We now review the properties of $S(\omega)$ for a complex, \mathcal{PT} -symmetric system [8]. (Henceforth, \mathcal{P} will refer to any linear symmetry operation such that $\mathcal{P}^2 = 1$, including not only parity, but also π rotations and inversion.) First, observe that the \mathcal{T} operator maps incoming channel modes to outgoing ones:

$$\mathcal{T} u_{\mu}^{\text{in}}(\vec{r}, \omega) = u_{\mu}^{\text{out}}(\vec{r}, \omega^*), \quad (4)$$

where \mathcal{T} is the antilinear complex conjugation operator. The generalized parity operator P is a linear operator acting on scalar fields satisfying

$$P u_{\mu}^{\text{in/out}}(\vec{r}, \omega) = \sum_{\nu} \mathcal{P}_{\nu\mu} u_{\nu}^{\text{in/out}}(\vec{r}, \omega), \quad (5)$$

where the system-dependent matrix \mathcal{P} mixes the channel functions, but never transforms between incoming and outgoing channels. Note that $(\mathcal{PT})^2 = 1$.

If the system is \mathcal{PT} symmetric, for any solution (2) there exists a valid solution $(\mathcal{PT})E(\vec{r})$ at frequency ω^* :

$$\sum_{\mu} [(\mathcal{PT}\varphi)_{\mu} u_{\mu}^{\text{in}}(\vec{r}, \omega^*) + (\mathcal{PT}\psi)_{\mu} u_{\mu}^{\text{out}}(\vec{r}, \omega^*)]. \quad (6)$$

Comparing this to (2) and (3), we conclude that

$$(\mathcal{PT})S(\omega^*)(\mathcal{PT}) = S^{-1}(\omega). \quad (7)$$

This is the fundamental relation obeyed by \mathcal{PT} -symmetric S matrices, and we will now show that it has important implications for the eigenvalue spectrum.

Multiplying both sides of (7) by an eigenvector ψ_n of $S(\omega)$ with eigenvalue s_n gives

$$S(\omega^*)(\mathcal{PT}\psi_n) = \frac{1}{s_n^*}(\mathcal{PT}\psi_n). \quad (8)$$

Hence, the inverse of the complex conjugate of any eigenvalue of $S(\omega)$ is an eigenvalue of $S(\omega^*)$. For real ω , this implies that $|\det S(\omega)| = 1$, just as for S matrices having pure \mathcal{T} symmetry, which are unitary. Unitarity imposes a stronger constraint: *each* eigenvalue is unimodular, so unitary S matrices do not have poles or zeros for real ω . Both \mathcal{T} and \mathcal{PT} symmetry imply that poles and zeros occur in complex conjugate pairs.

Symmetry-breaking transition.—Eq. (7) is a weaker constraint than unitarity, and can be satisfied in two ways: either each eigenvalue is itself unimodular, or the eigenvalues form pairs with reciprocal moduli. These two possibilities correspond to symmetric and symmetry-broken scattering behavior. In 1D, there are just two scattering eigenvectors, so the entire S matrix is either in the symmetric or broken-symmetry phase. (In higher dimensions, as we will see, the transition can occur in different eigenspaces of the S matrix, so that the system can be in a mixed phase; initially we focus on 1D.) Let us denote the eigenvectors by ψ_{\pm} . In the symmetric phase, each ψ_{\pm} is itself \mathcal{PT} symmetric, i.e. $\mathcal{PT}\psi_{\pm} \propto \psi_{\pm}$, so the eigenstate exhibits no net amplification nor dissipation ($|s_{\pm}| = 1$). In the broken-symmetry phase, ψ_{\pm} is not itself \mathcal{PT} symmetric but the *pair* satisfies \mathcal{PT} by transforming into each other: $\mathcal{PT}\psi_{\pm} = \psi_{\mp}$, where $s_{\pm} = 1/s_{\mp}^*$. Each eigenstate in the pair spontaneously breaks \mathcal{PT} symmetry; one exhibits amplification, and the other dissipation. Similar to transitions in Hamiltonian systems, such as the transition from real to complex eigenvalues of \mathcal{PT} -symmetric

Hamiltonians, the scattering transition can be induced by tuning the parameters of $S(\omega)$ [12].

Having shown that a \mathcal{PT} -breaking transition can take place in the scattering behavior of \mathcal{PT} -symmetric systems, we turn now to some concrete examples to see how it occurs. First, consider an arbitrary 1D system with \mathcal{PT} -symmetric $n(x)$; its S matrix is parameterized by

$$S = \begin{pmatrix} r_L & t \\ t & r_R \end{pmatrix} = t \begin{pmatrix} \frac{1-|t|^{-2}}{ib} & 1 \\ 1 & ib \end{pmatrix}, \quad (9)$$

where $b \equiv -ir_R/t \in \mathbb{R}$, r_R, r_L are the reflection amplitudes from right and left, and t is the (direction-independent) transmission amplitude. In general $|r_R| \neq |r_L|$, but their relative phase must be 0 or π . Although S depends on three real numbers, $\{\text{Re}(t), \text{Im}(t), b\}$, its eigenvalues only depend on two, $|t|$ and b , for we can scale out the phase of t . One can show that the criterion for the eigenvalues of S to be unimodular is:

$$|(r_L - r_R)/t| \equiv B(\omega, \tau) \leq 2. \quad (10)$$

For fixed $\text{Re}\{n(x)\}$, we write B as a function of ω and a \mathcal{T} -breaking parameter τ . On varying ω and/or τ , violating (10) brings us into the broken-symmetry phase.

Figure 1 shows how the transition occurs by varying ω , in a simple slab of total length L with fixed $n = n_0 \pm i\tau$ in each half. The critical frequency can be shown to be

$$\omega_c \approx \ln(2n_0/\tau)c/\tau L. \quad (11)$$

The resulting “phase diagram” is shown in Fig. 2(a). The discrete points in the broken-symmetry phase correspond to the CPA-laser solutions which we will soon discuss; these lie along a line given by the equation [7]

$$\omega \approx \omega_c + (c/\tau L) \ln[(n_0^2 + 1)/(n_0^2 - 1)]. \quad (12)$$

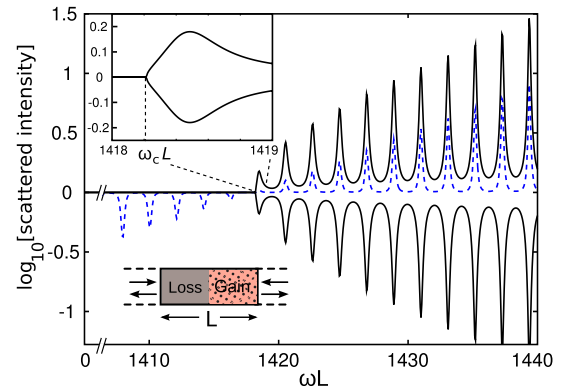


FIG. 1 (color online). Semilog plot of S -matrix eigenvalue intensities $\log_{10}|s_{\pm}|^2$ versus frequency ω (solid curves), for a 1D \mathcal{PT} -symmetric system of length L with balanced refractive index $n = 3 \pm 0.005i$ in each half (inset). The \mathcal{PT} symmetry is spontaneously broken at $\omega_c \approx 1418.21/L$. Dashed curves show the minimum scattered intensity for equal-intensity input beams of variable relative phase; in the broken-symmetry regime, net amplification always occurs.

The strong oscillations of $\log|s_{\pm}|$ in Fig. 1 occur as the system crosses this curve. Without fine-tuning n_0 , the system will not hit one of the CPA-laser solutions exactly; hence, varying ω does not bring $\log|s_{\pm}|$ to $\pm\infty$, though it can become quite large.

With different $n(x)$, more complicated phase diagrams are possible. Figure 2(b) shows the effect of a real-index region inserted between the gain and loss regions. As we increase ω , internal resonances in the real-index region cause the trapping time to oscillate with ω ; hence the effective \mathcal{T} -breaking perturbation strength oscillates, and the system reenters the symmetric phase periodically.

In these 1D structures, the symmetry-breaking transition can be probed by measuring the total intensity scattered by equal-intensity input beams of variable relative phase ϕ . In the symmetric phase, two particular values of $\phi(\omega, n)$ correspond to unimodular S -matrix eigenstates; for other values of ϕ , both net amplification and dissipation can be observed in the total output $|S\psi|^2$, since the system is not unitary. In the broken-symmetry phase, where $|s_+| > 1$, it can be shown that

$$|S\psi|^2 = 1 + |a|^2(|s_+|^2 + |s_+|^{-2} - 2) > 1, \quad (13)$$

where $a = \psi_1^T \psi / \psi_1^T \psi_1$. Thus all values of ϕ for balanced inputs ψ give rise to amplification (Fig. 1). This signature of the \mathcal{PT} transition should be easily observable.

Figure 3 shows the behavior of the poles and zeros in the complex plane for the simple 2-layer structure. When

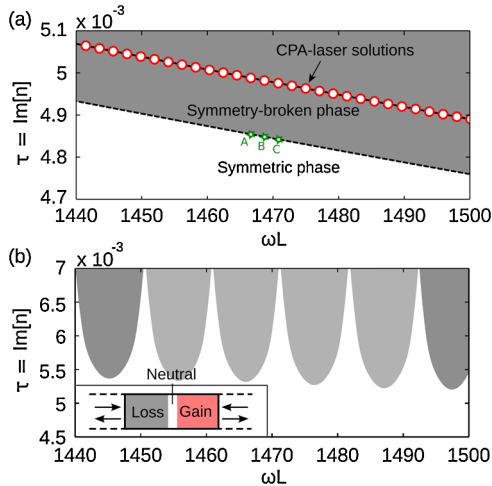


FIG. 2 (color online). Phase diagram of S -matrix eigenvalues for 1D \mathcal{PT} -symmetric systems. The structure in (a) is the same as in Fig. 1. The structure in (b) has a real-index region of length $L/10$ in the center, with all other parameters the same as in (a). It shows reentrant behavior as ω is varied. White areas show the \mathcal{PT} -symmetric phase, and grey areas show the \mathcal{PT} -broken phase; circles in (a) show the CPA-laser solutions. In (a), the dashed line shows the approximate phase boundary given by Eq. (11); the points A, B, and C on this boundary are referenced in Fig. 3. For fixed ω , stronger \mathcal{T} breaking always favors broken symmetry.

$\tau = 0$, the poles and zeros of the \mathcal{T} -symmetric S matrix are located symmetrically around the real axis, and can be labeled by the parity of the corresponding eigenvectors. As τ increases from zero, \mathcal{PT} symmetry requires that the zeros and poles move symmetrically in the complex plane. Neighboring zeros (poles) approach each other pairwise and undergo an anticrossing, causing strong parity mixing. The horizontal motion of the poles and zeros corresponds to the symmetric phase of the scatterer, and the vertical motion to the broken-symmetry phase. The anticrossing corresponds to the phase boundary, as can be seen by comparing the points labeled A, B, and C in Figs. 2 and 3.

CPA Laser.—The CPA-laser points occur when a pole and a zero of the S matrix coincide on the real axis, as shown in Fig. 3. This corresponds to the singular case in which $|s_n| \rightarrow 0$, implying $|1/s_n^*| \rightarrow \infty$, while their product remains unity. Physically, the scattering system behaves simultaneously as a laser at threshold and a CPA [9]. Figure 3 also indicates a very interesting property of these solutions: only half the zero-pole pairs flow upwards and reach the real axis, while the other half go off to infinity and do not produce laser-absorber modes; thus, the CPA-laser lines have *twice* the free spectral range of the passive cavity resonances. For $\omega L \gg 1$, they occur at frequencies $\omega_m \approx (2m + 0.5)\pi/(n_0L)$, where m is an integer, exactly halfway between alternate pairs of passive cavity resonances. This property of the CPA laser should be straightforward to demonstrate experimentally.

In a general \mathcal{PT} -symmetric system, the poles and zeros are related by complex conjugation, so any \mathcal{PT} system that is a laser *must* be a CPA laser. Even away from the CPA-laser singularities, the \mathcal{PT} -symmetric cavity in the

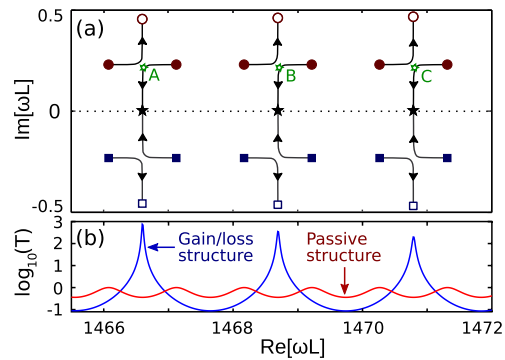


FIG. 3 (color online). (a) Trajectories of S -matrix poles and zeros for the two-layer system of Fig. 1. Filled circles and squares show the zeros and poles for $\tau = |\text{Im}[n]| = 0$; solid curves show their trajectories as τ is increased. Filled stars show CPA-laser solutions where zeros and poles meet on the real axis. For the middle set of trajectories, this happens at $\omega L = 1468.7$, $\tau = 4.982 \times 10^{-3}$. The transition occurs at the “anticrossings,” indicated by the representative points A, B, and C, which match the values of $\{\text{Re}[\omega], \tau\}$ marked in Fig. 2. (b) Log of the transmittance for $\tau = 0.005$ and $\tau = 0$, showing the doubled free spectral range in the CPA laser.

broken-symmetry phase is a unique interferometric amplifier: for coherent input radiation not corresponding closely to the damped S -matrix eigenvector, amplification takes place—in particular, this typically occurs for one-sided illumination. However, coherent illumination from both sides, with appropriate relative phase and amplitude corresponding to the damped eigenvector, leads instead to strong absorption. The CPA laser is the extreme case, where the gain or loss contrast is infinite.

The possibility of lasing in a \mathcal{PT} system is not obvious, as naively one might argue that photons traversing the device should experience no net gain. The presence of spontaneous symmetry breaking invalidates this argument. Photons in amplifying modes spend more time in the gain region than the loss region; hence, as the gain or loss parameter τ is increased, it eventually becomes possible for one mode to overcome the outcoupling loss and lase. As a self-organized *oscillator*, the CPA laser will automatically emit in the amplifying eigenstate.

Disk \mathcal{PT} scatterer.—Consider any 2D \mathcal{PT} -symmetric body in free space. Scattering in this system is naturally described using angular momentum channel functions,

$$u_m^{\text{in/out}}(r, \phi, \omega) = H_m^\pm(\omega r) e^{\pm im\phi}, \quad m = 0, \pm 1, \dots \quad (14)$$

For a region of linear dimensions R and average index \bar{n} , only states with $m \lesssim \bar{n}kR$ are significantly scattered, so we can truncate the infinite S matrix to $N \gtrsim \bar{n}kR$ channels. As this S matrix is tuned, multiple pairs of eigenstates can undergo \mathcal{PT} breaking, so the phases of the S matrix are indexed by the number of pairs of \mathcal{PT} -broken eigenstates. In Fig. 4, we show this behavior for a disk with balanced semicircular gain and loss regions, with the S matrix calculated using the R -matrix method [10,13]. The symmetry breaking is nonmonotonic in ω , similar to the 1D example of Fig. 2(b). The regions of strong \mathcal{PT} breaking are roughly associated with the resonances of the

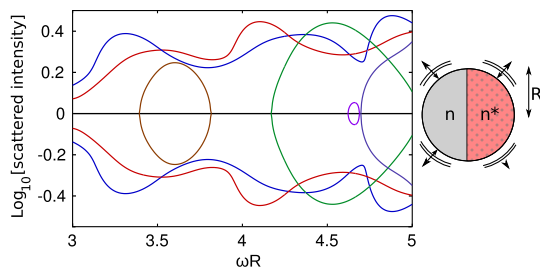


FIG. 4 (color online). Semilog plot of S -matrix eigenvalue intensities $\log_{10}|s|^2$ versus frequency ω , for a 2D \mathcal{PT} -symmetric disk of radius R and refractive indices $1.5 \pm 0.1i$. The S matrix is truncated to angular momenta $|m| \leq 20$. Most of the 41 eigenvalues are unimodular at these frequencies; those with higher average m , near resonances of the passive disk, show greater symmetry breaking. The colors distinguish several different pairs of \mathcal{PT} -broken eigenvalues.

\mathcal{T} -symmetric disk, as symmetry breaking is enhanced by the dwell time in the medium. This points to the possibility of microdisk based amplifier absorbers.

Conclusion.—We have shown that \mathcal{PT} -symmetric optical scattering systems generically display spontaneous symmetry breaking, with a unimodular phase where the S -matrix eigenstates are norm preserving, and a broken-symmetry phase in which they are pairwise amplifying and damping with reciprocal eigenvalue moduli. This \mathcal{PT} -breaking transition can be tested experimentally, using 1D heterojunction geometries (with realistic values of the gain or loss parameter) that are distinctly different from the paraxial geometries previously suggested for observing \mathcal{PT} symmetry breaking in optics [3–6]. In our analysis, we have neglected the role of the noise due to amplified spontaneous emission, which may be significant at the singular CPA-laser points [8]. However this noise should not preclude observation of experimental signatures of CPA lasing, e.g., the doubling of the free spectral range relative to the passive cavity. Moreover one may study the interferometric amplifying behavior in the broken-symmetry phase well below the CPA-laser points, where the noise will be substantially smaller.

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