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2×2 $\mathcal{PT}$-symmetric matrices and their applications

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Two formulations for constructing a non-Hermitian matrix with all real eigenvalues are studied. They are called $\mathcal{PT}$ symmetry and pseudo-Hermiticity in the literature. Explicit 2×2 matrices of both forms are provided. They are characterized by six real parameters and are hence more general than Hermitian matrices. The equivalence of the two formulations is established. A 2×2 matrix with all real eigenvalues is $\mathcal{PT}$-symmetric and pseudo-Hermitian at the same time. The application in time-dependent problems is discussed and a new geometry phase is obtained.

1. Introduction

Non-Hermitian matrices with all real eigenvalues often appear in the study of $\mathcal{PT}$-symmetric quantum mechanics [1,2]. In particular, 2×2 matrices are commonly used [3]. In this paper, we classify all 2×2 matrices with all real or complex conjugate pair eigenvalues. We use two formulations: $\mathcal{PT}$ symmetry and pseudo-Hermiticity.

In both formulations, we allow the Hamiltonian in a *non-dissipative* two-level quantum system to take a non-Hermitian matrix form. 'Non-dissipative' means that the eigenvalues of the Hamiltonian are real. 'Non-Hermitian matrix' is a matrix with $H \neq H^\dagger$, where $\dagger$ stands for complex conjugation and transpose.

The Hilbert space associated with such a (diagonalizable) Hamiltonian requires a non-trivial metric operator [4]

$$(\cdot, \cdot) \equiv \langle \cdot | W | \cdot \rangle, \tag{1.1}$$

where $\langle \cdot | = | \cdot \rangle^\dagger$ and W is a Hermitian matrix with all positive eigenvalues. The Hamiltonian H is self-adjoint with respect to this inner-product,

$$(\cdot, H\cdot) = (H\cdot, \cdot). \tag{1.2}$$

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This condition leads to a matrix equation between H and W,

$$WH = H^{\dagger}W. \tag{1.3}$$

One may take this equation as the definition of the metric operator W.[1] In general, W depends on the Hamiltonian H, i.e. it is dynamical. In conventional quantum mechanics, the Hamiltonian is a Hermitian matrix. In this case, W can be chosen as the unit matrix and it is no longer dynamical.

In §2, we study $\mathcal{PT}$-symmetric matrices by defining the parity operator as a real involution. A 2×2 $\mathcal{PT}$-symmetric matrix is characterized by six real parameters, two more than a Hermitian matrix with the same dimension. When all the eigenvalues of a $\mathcal{PT}$-symmetric matrix are real, the eigenstates are also eigenstates of $\mathcal{PT}$.

In §3, we study the $\tilde{\mathcal{P}}$-pseudo-Hermitian matrices, where the $\tilde{\mathcal{P}}$ operator is a Hermitian involution. The $\tilde{\mathcal{P}}$ operator defines a Pontryagin space with indefinite norms. A 2×2 $\tilde{\mathcal{P}}$-pseudo-Hermitian matrix is also characterized by six real parameters. A Hermitian matrix can always be considered as a special case of $\tilde{\mathcal{P}}$-pseudo-Hermitian matrices.

In §4, we discuss the relation between the two formulations. A real-symmetric involution can serve as the parity in $\mathcal{PT}$ symmetry and the $\tilde{\mathcal{P}}$ operator in pseudo-Hermiticity. A complex-symmetric matrix can be $\mathcal{PT}$-symmetric and $\mathcal{P}$-pseudo-Hermitian with the *same* $\mathcal{P}$. In the case of 2×2, a $\mathcal{PT}$-symmetric matrix is always $\tilde{\mathcal{P}}$-pseudo-Hermitian with respect to some $\tilde{\mathcal{P}}$ operator, and vice versa, although in general $\mathcal{P} \neq \tilde{\mathcal{P}}$.

In §5, we apply our parametrization to a time-dependent two-level quantum system. We find new results in a geometry-phase problem with a tunable fictitious magnetic monopole and a measurable Dirac string.

Finally, we give some concluding remarks in §6.

2. $\mathcal{PT}$ symmetry

To construct $\mathcal{PT}$-symmetric matrices, let us start with the definitions of the parity operator $\mathcal{P}$ and the time reversal operator $\mathcal{T}$. The time reversal operator is an anti-linear operator and it is an involution. Without loss of generality, we define $\mathcal{T}$ simply as the complex conjugation operator

$$\mathcal{T} \equiv * \;\Rightarrow\; \mathcal{T}^2 = \mathbf{1}. \tag{2.1}$$

That is, for a matrix (operator), $\mathcal{T}A\mathcal{T} = A^*$, and for a vector (state), $\mathcal{T}|\psi\rangle = |\psi\rangle^*$, where $*$ represents complex conjugation. We define the parity operator as a real involutory matrix [5]

$$\mathcal{P} = \mathcal{P}^* \quad \text{and} \quad \mathcal{P}^2 = \mathbf{1}. \tag{2.2}$$

As usual the time reversal and the parity commute: $[\mathcal{P}, \mathcal{T}] = 0$.

A matrix Hamiltonian H is said to have $\mathcal{PT}$ symmetry if it commutes with the combination of $\mathcal{PT}$

$$[\mathcal{PT}, H] = 0 \;\Leftrightarrow\; \mathcal{P}H = H^*\mathcal{P}. \tag{2.3}$$

It is easy to verify that all the eigenvalues of H either are real or form a complex conjugate pair. If a $\mathcal{PT}$-symmetric matrix Hamiltonian H has an entirely real spectrum, we say that the $\mathcal{PT}$ symmetry is *unbroken*. In this case, all the eigenstates of H are also eigenstates of $\mathcal{PT}$.

All the parity operators with the same eigenvalues can be linked by a real similarity transformation:

$$\mathcal{P} = R\mathcal{P}_0R^{-1}, \quad \text{with } R = R^*. \tag{2.4}$$

Without loss of generality, we start with a simple diagonal parity operator

$$\mathcal{P}_0 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.5}$$

[1]In general, there are multiple solutions for W, e.g. W_0 in (2.10) and $\tilde{W}_0$ in (3.9). Therefore, equation (1.3) defines a set of possible W.

A $\mathcal{P}_0\mathcal{T}$-symmetric matrix Hamiltonian can be parametrized as

$$\begin{aligned} H_0 &= \epsilon\sigma_0 + (\mathrm{i}\rho, \gamma\sin\delta, \gamma\cos\delta)\cdot\boldsymbol{\sigma} \\ &= \begin{pmatrix} \epsilon+\gamma\cos\delta & -\mathrm{i}(\gamma\sin\delta-\rho) \\ \mathrm{i}(\gamma\sin\delta+\rho) & \epsilon-\gamma\cos\delta \end{pmatrix} \end{aligned} \tag{2.6}$$

with four real parameters: ϵ, γ, ρ, and δ. Here $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are Pauli matrices and σ_0 is the unit matrix. The eigenvalues of H_0 are

$$E_\pm = \epsilon \pm \sqrt{\gamma^2-\rho^2}. \tag{2.7}$$

They are real when $\gamma^2 \geq \rho^2$. This is the condition for $\mathcal{PT}$ symmetry to not be broken. The corresponding eigenvectors are

$$|E_\pm\rangle = n_\pm \begin{pmatrix} \gamma\mathrm{e}^{\mathrm{i}\delta} - \mathrm{i}\rho \pm \sqrt{\gamma^2-\rho^2} \\ \gamma\mathrm{e}^{\mathrm{i}\delta} + \mathrm{i}\rho \mp \sqrt{\gamma^2-\rho^2} \end{pmatrix}, \tag{2.8}$$

where $n_\pm$ are the normalization constants. It can be checked that $|E_\pm\rangle$ are also eigenvectors of $\mathcal{P}_0\mathcal{T}$ with the eigenvalues of modulus one:

$$\mathcal{P}_0\mathcal{T}|E_\pm\rangle = \mathrm{e}^{\mathrm{i}\alpha_\pm}|E_\pm\rangle. \tag{2.9}$$

We can always choose the phases of the eigenvectors such that the eigenvalues of $\mathcal{P}_0\mathcal{T}$ are one.

Solving the self-adjoint condition in (1.3), we get the form of the metric operator

$$\begin{aligned} W_0 &= u[\gamma\sigma_0 + (0, v\sin\delta-\rho\cos\delta, v\cos\delta+\rho\sin\delta)\cdot\boldsymbol{\sigma}] \\ &= u\begin{pmatrix} \gamma+(v\cos\delta+\rho\sin\delta) & -\mathrm{i}(v\sin\delta-\rho\cos\delta) \\ \mathrm{i}(v\sin\delta-\rho\cos\delta) & \gamma-(v\cos\delta+\rho\sin\delta) \end{pmatrix}, \end{aligned} \tag{2.10}$$

where u and v are arbitrary real constants with the constraints $u\gamma > 0$ and $v^2 < \gamma^2 - \rho^2$. With this metric operator, the eigenvectors of H_0 are orthogonal

$$\langle E_+|W_0|E_-\rangle = 0 = \langle E_-|W_0|E_+\rangle \tag{2.11}$$

and normalized to

$$\mathcal{N}_\pm \equiv \langle E_\pm|W_0|E_\pm\rangle = 4|n_\pm|^2 u\gamma\sqrt{\gamma^2-\rho^2}\left(\sqrt{\gamma^2-\rho^2} \pm v\right). \tag{2.12}$$

Given u and v, one can always normalize the eigenvectors by choosing proper $n_\pm$. It is also true in the reversed way: for arbitrary non-vanishing $n_\pm$, one can always tune u and v in the metric to normalize the eigenvectors.

The real matrices generating non-trivial parity operators can be parametrized either by

$$R_1(\theta,\varphi) = \mathrm{e}^{-\varphi\sigma_3/2}\mathrm{e}^{-\mathrm{i}\theta\sigma_2/2} \tag{2.13}$$

or by

$$R_2(\theta,\varphi) = \mathrm{e}^{-\varphi\sigma_3/2}\mathrm{e}^{-\theta\sigma_1/2}. \tag{2.14}$$

Note that they are not equivalent. According to these transformations, a generic parity operator has the form

$$\mathcal{P}_1 = R_1(\theta,\varphi)\mathcal{P}_0R_1^{-1}(\theta,\varphi) = \begin{pmatrix} \cos\theta & \sin\theta\,\mathrm{e}^{-\varphi} \\ \sin\theta\,\mathrm{e}^{\varphi} & -\cos\theta \end{pmatrix} \tag{2.15}$$

or

$$\mathcal{P}_2 = R_2(\theta,\varphi)\mathcal{P}_0R_2^{-1}(\theta,\varphi) = \begin{pmatrix} \cosh\theta & \sinh\theta\,\mathrm{e}^{-\varphi} \\ -\sinh\theta\,\mathrm{e}^{\varphi} & -\cosh\theta \end{pmatrix}. \tag{2.16}$$

The same transformation preserves the property of $\mathcal{PT}$ symmetry. That is, if H_0 is $\mathcal{P}_0\mathcal{T}$-symmetric, then

$$H = RH_0R^{-1} \tag{2.17}$$

is $\mathcal{PT}$-symmetric:

$$\mathcal{P}_0 H_0 = H_0^* \mathcal{P}_0 \;\Leftrightarrow\; \mathcal{P}H = H^*\mathcal{P}. \tag{2.18}$$

Two parametrizations of $\mathcal{PT}$-symmetric matrix Hamiltonians can be obtained by the corresponding transformations [5]:

$$\begin{aligned} H_1 &= R_1(\theta,\varphi) H_0 R_1^{-1}(\theta,\varphi) \\ &= \begin{pmatrix} \epsilon + \gamma\cos\delta\cos\theta - \mathrm{i}\rho\sin\theta & (\gamma\cos\delta\sin\theta - \mathrm{i}\gamma\sin\delta + \mathrm{i}\rho\cos\theta)\mathrm{e}^{-\varphi} \\ (\gamma\cos\delta\sin\theta + \mathrm{i}\gamma\sin\delta + \mathrm{i}\rho\cos\theta)\mathrm{e}^{\varphi} & \epsilon - \gamma\cos\delta\cos\theta + \mathrm{i}\rho\sin\theta \end{pmatrix} \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} H_2 &= R_2(\theta,\varphi) H_0 R_2^{-1}(\theta,\varphi) \\ &= \begin{pmatrix} \epsilon + \gamma\cos(\delta + \mathrm{i}\theta) & -\mathrm{i}[\gamma\sin(\delta + \mathrm{i}\theta) - \rho]\mathrm{e}^{-\varphi} \\ \mathrm{i}[\gamma\sin(\delta + \mathrm{i}\theta) + \rho]\mathrm{e}^{\varphi} & \epsilon - \gamma\cos(\delta + \mathrm{i}\theta) \end{pmatrix}. \end{aligned} \tag{2.20}$$

Note that either parametrization has six parameters, four from the original H_0 and two from the transformation R_1 or R_2. When the parity is transformed to $\mathcal{P} = R\mathcal{P}_0 R^{-1}$, the Hamiltonian changes to $H = RH_0R^{-1}$. The metric operator transforms accordingly,

$$W = (R^{-1})^\dagger W_0 R^{-1}. \tag{2.21}$$

3. Pseudo-Hermiticity

Pseudo-Hermiticity is another formulation for constructing non-Hermitian matrices with all real eigenvalues [6]. This time, let us start with a Hermitian involutory, i.e. unitary operator $\tilde{\mathcal{P}}$

$$\tilde{\mathcal{P}} = \tilde{\mathcal{P}}^\dagger \quad \text{and} \quad \tilde{\mathcal{P}}^2 = \mathbf{1}. \tag{3.1}$$

A $\tilde{\mathcal{P}}$-pseudo-Hermitian matrix Hamiltonian $\tilde{H}$ satisfies

$$\tilde{\mathcal{P}}\tilde{H}\tilde{\mathcal{P}} = \tilde{H}^\dagger \;\Leftrightarrow\; \tilde{\mathcal{P}}\tilde{H} = \tilde{H}^\dagger\tilde{\mathcal{P}}. \tag{3.2}$$

Clearly, in this formulation, the $\tilde{\mathcal{P}}$ operator plays the role of the metric operator except that $\tilde{\mathcal{P}}$ may have negative eigenvalues. The linear vector space associated with an inner-product defined by an indefinite metric operator is called a Pontryagin space. It is a finite-dimensional version of the Krein space.

A unitary transformation preserves the property of the $\tilde{\mathcal{P}}$ operator defined in (3.1)

$$\tilde{\mathcal{P}} = U\mathcal{P}_0 U^\dagger, \quad \text{with } U^{-1} = U^\dagger. \tag{3.3}$$

The same transformation transforms the Hamiltonian and the metric operator properly

$$H = UH_0U^\dagger \quad \text{and} \quad W = UW_0U^\dagger. \tag{3.4}$$

Just like the method used in §2, we may start with a simple operator $\tilde{\mathcal{P}}_0 = \sigma_3$. A $\tilde{\mathcal{P}}_0$-pseudo-Hermitian Hamiltonian has the form

$$\tilde{H}_0 = \epsilon\sigma_0 + (\mathrm{i}\rho\sin\delta, \mathrm{i}\rho\cos\delta, \gamma)\cdot\boldsymbol{\sigma} = \begin{pmatrix} \epsilon + \gamma & \rho\mathrm{e}^{\mathrm{i}\delta} \\ -\rho\mathrm{e}^{-\mathrm{i}\delta} & \epsilon - \gamma \end{pmatrix}. \tag{3.5}$$

The eigenvalues of $\tilde{H}_0$ are

$$E_\pm = \epsilon \pm \sqrt{\gamma^2 - \rho^2}. \tag{3.6}$$

They are real when $\gamma^2 \geq \rho^2$. The eigenvectors are

$$|E_\pm\rangle = n_\pm \begin{pmatrix} \gamma \pm \sqrt{\gamma^2 - \rho^2} \\ -\rho\mathrm{e}^{-\mathrm{i}\delta} \end{pmatrix}, \tag{3.7}$$

where $n_\pm$ are the normalization constants. The two eigenvectors are orthogonal with respect to the $\tilde{\mathcal{P}}_0$-inner-product

$$\langle E_+|\tilde{\mathcal{P}}_0|E_-\rangle = 0 = \langle E_-|\tilde{\mathcal{P}}_0|E_+\rangle. \tag{3.8}$$

However, one of the $\tilde{\mathcal{P}}_0$-norms is negative. To define the Hilbert space, one needs to solve the self-adjoint condition in (1.3) to get the positive definite metric operator $\tilde{W}_0$. The solution is

$$\tilde{W}_0 = u[\gamma\sigma_0 + (\rho\cos\delta, -\rho\sin\delta, v)\cdot\boldsymbol{\sigma}] = u\begin{pmatrix}\gamma+v & \rho e^{i\delta}\\ \rho e^{-i\delta} & \gamma-v\end{pmatrix}, \tag{3.9}$$

where u and v are arbitrary constants with the constraints $u\gamma > 0$ and $v^2 < \gamma^2 - \rho^2$. The eigenvalues of $\tilde{W}_0$ are

$$\omega_\pm = u(\gamma \pm \sqrt{\rho^2 + v^2}). \tag{3.10}$$

They are always positive. With this metric operator, the eigenvectors of $\tilde{H}_0$ with different eigenvalues are orthogonal

$$\langle E_+|\tilde{W}_0|E_-\rangle = 0 = \langle E_-|\tilde{W}_0|E_+\rangle \tag{3.11}$$

and the norms are

$$\mathcal{N}_\pm \equiv \langle E_\pm|\tilde{W}_0|E_\pm\rangle = 2|n_\pm|^2 u\sqrt{\gamma^2-\rho^2}(\gamma \pm \sqrt{\gamma^2-\rho^2})(\sqrt{\gamma^2-\rho^2} \pm v). \tag{3.12}$$

Given u and v, one can always normalize the eigenvectors by choosing $n_\pm$ properly. It is also true that for arbitrary non-vanishing $n_\pm$, one can always tune u and v in the metric to normalize the eigenvectors.

A generic unitary matrix for generating a non-trivial $\tilde{\mathcal{P}}$ operator can be parametrized as

$$U(\theta,\varphi) = e^{-i\varphi\sigma_3/2}e^{-i\theta\sigma_2/2}. \tag{3.13}$$

It transforms $\tilde{\mathcal{P}}_0$ to

$$\tilde{\mathcal{P}} = U(\theta,\varphi)\tilde{\mathcal{P}}_0 U^\dagger(\theta,\varphi) = \boldsymbol{n}^r\cdot\boldsymbol{\sigma} = \begin{pmatrix}\cos\theta & \sin\theta\, e^{-i\varphi}\\ \sin\theta\, e^{i\varphi} & -\cos\theta\end{pmatrix}, \tag{3.14}$$

where $\boldsymbol{n}^r \equiv (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ is a unit vector. The $\tilde{\mathcal{P}}$-pseudo-Hermitian Hamiltonian can be obtained by the same unitary transformation

$$\tilde{H} = U(\theta,\varphi)\tilde{H}_0 U^\dagger(\theta,\varphi) = \epsilon\sigma_0 + (\gamma\,\boldsymbol{n}^r + i\rho\sin\delta\,\boldsymbol{n}^\theta + i\rho\cos\delta\,\boldsymbol{n}^\varphi)\cdot\boldsymbol{\sigma}, \tag{3.15}$$

where $\boldsymbol{n}^\theta \equiv (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)$ and $\boldsymbol{n}^\varphi \equiv (-\sin\varphi, \cos\varphi, 0)$ are two unit vectors perpendicular to $\boldsymbol{n}^r$. The essentially same parametrization was first introduced in [7].[2] There are again six parameters, four from the $\tilde{H}_0$ and two from the U. Note that all 2×2 Hermitian matrices are special cases of this $\tilde{\mathcal{P}}$-pseudo-Hermitian matrix $\tilde{H}$ with $\rho = 0$.

The metric operator $\tilde{W}$ transforms the same way when one applies the unitary transformation to a $\tilde{\mathcal{P}}$-pseudo-Hermitian matrix Hamiltonian $\tilde{H}$

$$\tilde{W} = U(\theta,\varphi)\tilde{W}_0 U^\dagger(\theta,\varphi) = u[\gamma\sigma_0 + (v\,\boldsymbol{n}^r + \rho\cos\delta\,\boldsymbol{n}^\theta - \rho\sin\delta\,\boldsymbol{n}^\varphi)\cdot\boldsymbol{\sigma}]. \tag{3.16}$$

4. Relation between $\mathcal{PT}$ symmetry and pseudo-Hermiticity

Obviously a real-symmetric parity operator is also Hermitian. It can serve both as the parity in $\mathcal{PT}$ symmetry and as the indefinite metric in pseudo-Hermiticity. The intersection set between real similarity transformations and unitary transformations is the set of orthogonal transformations. The resulting $\mathcal{PT}$-symmetric and $\mathcal{P}$-pseudo-Hermitian (with the *same* $\mathcal{P}$) matrix is complex-symmetric. This narrow intersection is studied in [8].

[2]It was called the general $\mathcal{PT}$-symmetric matrix Hamiltonian in [7]. Here, we follow the terminology used in [6] and call it a pseudo-Hermitian Hamiltonian.

Now we have a more general parity operator and more parameters in matrix Hamiltonians. One may ask a slightly different question: Whether a $\mathcal{PT}$-symmetric matrix is $\tilde{\mathcal{P}}$-pseudo-Hermitian with respect to a *different* $\tilde{\mathcal{P}}$? Or the reversed question: Whether a $\tilde{\mathcal{P}}$-pseudo-Hermitian matrix is $\mathcal{PT}$-symmetric? In the case of 2×2, the answer to both questions is yes. The two formulations are equivalent: for any $\mathcal{PT}$-symmetric matrix H, one can always find a $\tilde{\mathcal{P}}$ operator such that H is $\tilde{\mathcal{P}}$-pseudo-Hermitian; for any $\tilde{\mathcal{P}}$-pseudo-Hermitian matrix $\tilde{H}$, one can always find a parity operator $\mathcal{P}$ such that $\tilde{H}$ is $\mathcal{PT}$-symmetric.

For example, the $\mathcal{P}_0\mathcal{T}$-symmetric H_0 in (2.6) is $\tilde{\mathcal{P}}$-pseudo-Hermitian with respect to

$$\tilde{\mathcal{P}} = \pm(0, \sin\delta, \cos\delta) \cdot \boldsymbol{\sigma} = \pm \begin{pmatrix} \cos\delta & -\mathrm{i}\sin\delta \\ \mathrm{i}\sin\delta & -\cos\delta \end{pmatrix}. \tag{4.1}$$

The $\tilde{\mathcal{P}}_0$-pseudo-Hermitian $\tilde{H}_0$ in (3.5) is $\mathcal{PT}$-symmetric with

$$\mathcal{P} = \frac{\pm 1}{\sqrt{\gamma^2 - \rho^2\cos^2\delta}} \begin{pmatrix} \gamma & \rho\cos\delta \\ -\rho\cos\delta & -\gamma \end{pmatrix}. \tag{4.2}$$

The generic mapping between the parity $\mathcal{P}$ in $\mathcal{PT}$ symmetry and the $\tilde{\mathcal{P}}$ operator in pseudo-Hermiticity for a given matrix Hamiltonian can be found in [5].

5. Applications

In conventional quantum mechanics, the simple 2×2 matrices serve as important toy models to study various quantum properties. We expect the same things in $\mathcal{PT}$-symmetric non-Hermitian quantum mechanics. The first 2×2 $\mathcal{PT}$-symmetric matrix Hamiltonian introduced in [3] was used to construct a proper inner-product with positive-definite norms. The same matrix was also used to discuss quantum entanglement [9,10]. A more general 2×2 matrix constructed in [8] was used to demonstrate the quantum brachistochrone phenomenon in a non-Hermitian system, which later caused a heated debate [11–13]. The most general 2×2 $\tilde{\mathcal{P}}$-pseudo-Hermitian matrix in (3.15) was introduced in [7]. It was used in the study of random matrix theory [14]. Here, we review another application of it in time-dependent $\mathcal{PT}$-symmetric quantum mechanics. We find that there is a non-conventional geometry phase in a time-dependent 2×2 system [15].

Suppose that a $\mathcal{PT}$-symmetric (or equivalently $\tilde{\mathcal{P}}$-pseudo-Hermitian) Hamiltonian H is parametrized by a multi-dimensional vector $\mathbf{X}$, which may slowly change in time $\mathbf{X} = \mathbf{X}(t)$. Because the metric operator depends on the Hamiltonian, in general it is also time-dependent:

$$W[\mathbf{X}(t)]H[\mathbf{X}(t)] = H^\dagger[\mathbf{X}(t)]W[\mathbf{X}(t)]. \tag{5.1}$$

One consequence of the time-dependent metric is that the evolution equation may not be the Schrödinger equation. Instead, in $\mathcal{PT}$-symmetric quantum mechanics, the time evolution is governed by the following equation (setting $\hbar = 1$ throughout) [15]:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\Psi(t)\rangle = \Lambda(t)|\Psi(t)\rangle, \quad \text{with } \Lambda(t) \equiv H[\mathbf{X}(t)] - \mathrm{i}K[\mathbf{X}(t)]. \tag{5.2}$$

Imposing the unitary condition, we get the following equation for $K[\mathbf{X}(t)]$:

$$\nabla W[\mathbf{X}(t)] \cdot \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} = K^\dagger[\mathbf{X}(t)]W[\mathbf{X}(t)] + W[\mathbf{X}(t)]K[\mathbf{X}(t)], \tag{5.3}$$

where ∇ is the gradient in the $\mathbf{X}$-space.

To find an interesting sample solution to (5.3), we put forward below an additional condition $K[\mathbf{X}(t)] = K^\dagger[\mathbf{X}(t)]$. We rewrite K as $K[\mathbf{X}(t)] = \mathbf{M}[\mathbf{X}(t)] \cdot (\mathrm{d}\mathbf{X}/\mathrm{d}t)$, with

$$\nabla W(\mathbf{X}) = \mathbf{M}(\mathbf{X})W(\mathbf{X}) + W(\mathbf{X})\mathbf{M}(\mathbf{X}) \tag{5.4}$$

and $\mathbf{M}(\mathbf{X}) = \mathbf{M}^\dagger(\mathbf{X})$.

The instantaneous eigenstates of $H[\mathbf{X}(t)]$ satisfy

$$H[\mathbf{X}(t)]|\psi_n[\mathbf{X}(t)]\rangle = E_n[\mathbf{X}(t)]|\psi_n[\mathbf{X}(t)]\rangle, \tag{5.5}$$

with the orthonormal condition $\langle\psi_n[\mathbf{X}(t)]|W[\mathbf{X}(t)]|\psi_m[\mathbf{X}(t)]\rangle = \delta_{mn}$. We can expand $|\Psi(t)\rangle$ as

$$|\Psi(t)\rangle = \sum_n c_n(t)\mathrm{e}^{\mathrm{i}\alpha_n(t)}|\psi_n[\mathbf{X}(t)]\rangle, \tag{5.6}$$

where c_n is the expansion coefficient, and α_n is a dynamical phase defined as the time-integral of the instantaneous real energy eigenvalue $E_n[\mathbf{X}(t)]$, i.e. $\alpha_n(t) \equiv -\int_0^t E_n[\mathbf{X}(t')]\mathrm{d}t'$. Substituting (5.6) into (5.2), one obtains

$$\dot{c}_m(t) = -c_m(t)\mathcal{G}_m(t) - \sum_{n\neq m} c_n(t)\mathrm{e}^{\mathrm{i}[\alpha_n(t)-\alpha_m(t)]}\mathcal{T}_{mn}, \tag{5.7}$$

where

$$\mathcal{G}_m(t) \equiv \{\langle\psi_m(\mathbf{X})|W(\mathbf{X})|\nabla\psi_m(\mathbf{X})\rangle + \langle\psi_m(\mathbf{X})|W(\mathbf{X})\mathbf{M}(\mathbf{X})|\psi_m(\mathbf{X})\rangle\} \cdot \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} \tag{5.8}$$

and

$$\mathcal{T}_{mn}(t) \equiv \left\{\langle\psi_m(\mathbf{X})|W(\mathbf{X})\left[\frac{\nabla H(\mathbf{X})}{E_n(\mathbf{X}) - E_m(\mathbf{X})}\right]|\psi_n(\mathbf{X})\rangle + \langle\psi_m(\mathbf{X})|W(\mathbf{X})\mathbf{M}(\mathbf{X})|\psi_n(\mathbf{X})\rangle\right\} \cdot \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} \tag{5.9}$$

with $\mathbf{X} = \mathbf{X}(t)$. Now suppose $\mathbf{X}(t)$ varies very slowly in time, such that the following adiabatic conditions hold:

$$\frac{\nabla H(\mathbf{X})}{[E_n(\mathbf{X}) - E_m(\mathbf{X})]^2} \cdot \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} \ll 1 \quad \text{and} \quad \frac{\mathbf{M}(\mathbf{X})}{E_n(\mathbf{X}) - E_m(\mathbf{X})} \cdot \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} \ll 1. \tag{5.10}$$

Then under an adiabatic approximation, the $\mathcal{G}_m$ term in (5.7) gives the solution $c_m(t) = c_m(0)\mathrm{e}^{\mathrm{i}\beta(t)}$. Because $\beta(t)$ is solely determined by the geometry of a navigation path in the $\mathbf{X}$-space and does not depend on the duration of the adiabatic process, it is of a geometric origin. In particular, for a cyclic path, this geometric phase becomes a Berry-like phase β_{B}, i.e.

$$\beta_{\mathrm{B}} = \mathrm{i}\oint[\langle\psi_m(\mathbf{X})|W(\mathbf{X})|\nabla\psi_m(\mathbf{X})\rangle + \langle\psi_m(\mathbf{X})|W(\mathbf{X})\mathbf{M}(\mathbf{X})|\psi_m(\mathbf{X})\rangle] \cdot \mathrm{d}\mathbf{X}. \tag{5.11}$$

Let us now apply these general considerations to a two-dimensional Hilbert space. Because $\mathcal{PT}$ symmetry is equivalent to $\tilde{\mathcal{P}}$ pseudo-Hermiticity, we use the generic matrix Hamiltonian $\tilde{H}$ in (3.15). For simplicity, we only let parameters θ and φ vary in time and keep others fixed. In the metric operator in (3.16), we choose $u = \mathrm{sign}(\gamma)/\sqrt{\gamma^2 - \rho^2}$ and $v = 0$.

In the (θ, ϕ) manifold, ϵ, γ, ρ and δ are constants and we obtain a simple Hermitian solution to (5.4):

$$\mathbf{M}_{2\times 2}(\mathbf{X}) = \tfrac{1}{2}\nabla\boldsymbol{\xi}(\mathbf{X}) \cdot \boldsymbol{\sigma}, \tag{5.12}$$

where

$$\boldsymbol{\xi} \equiv \frac{\rho}{\gamma}(\sin\delta\, \boldsymbol{n}^{\theta} - \cos\delta\, \boldsymbol{n}^{\varphi}). \tag{5.13}$$

After a lengthy calculation, we arrive at a rather compact expression for the geometric phase. For example, associated with $|E_+\rangle$, the Berry-like phase denoted β_{B}^{+} is found to be

$$\beta_{\mathrm{B}}^{+} = \oint[F_\theta(\mathbf{X})\mathrm{d}\theta + F_\varphi(\mathbf{X})\mathrm{d}\varphi], \tag{5.14}$$

where

$$F_\theta(\mathbf{X}) = \frac{\rho\cos\delta}{2[\gamma + \rho\sin\delta\sin\theta - \sqrt{\gamma^2 - \rho^2}\cos\theta]} \tag{5.15}$$

and

$$F_\varphi(\mathbf{X}) = \frac{1}{2\gamma}(\gamma + \sqrt{\gamma^2 - \rho^2}\cos\theta). \tag{5.16}$$

To better understand γ_B^+ we now treat θ and φ as two angular coordinates on a sphere defined by $\boldsymbol{r} \equiv r\boldsymbol{n}^r$. If we define

$$\boldsymbol{A} \equiv \frac{F_\theta(\mathbf{X})}{r}\boldsymbol{n}^\theta + \frac{F_\varphi(\mathbf{X})}{r\sin\theta}\boldsymbol{n}^\varphi, \tag{5.17}$$

then because $F_\theta(\mathbf{X})\mathrm{d}\theta + F_\varphi(\mathbf{X})\mathrm{d}\varphi = \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{r}$, equation (5.14) becomes

$$\beta_B^+ = \oint \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{r}. \tag{5.18}$$

Thus, we may regard $\boldsymbol{A}$ as a three-dimensional vector potential in the $\boldsymbol{r}$-space. With a fictitious magnetic field denoted $\boldsymbol{B}_{\mathrm{eff}}$, β_B^+ for an arbitrary closed path on the sphere can now be expressed as a magnetic flux through a surface enclosed by the closed path, i.e. $\beta_B^+ = \int\int \boldsymbol{B}_{\mathrm{eff}}\cdot\mathrm{d}\boldsymbol{S}$, where

$$\boldsymbol{B}_{\mathrm{eff}} = \left(1 + \frac{\sqrt{\gamma^2-\rho^2}}{\gamma}\right)\pi\,\delta(x)\delta(y)\boldsymbol{n}^z - \frac{\sqrt{\gamma^2-\rho^2}}{2\gamma}\frac{\boldsymbol{r}}{r^3}, \tag{5.19}$$

with $\boldsymbol{n}^z \equiv \boldsymbol{n}^r(\theta=0)$. $\boldsymbol{B}_{\mathrm{eff}}$ represents a fictitious singular field pointing at the north pole, plus a virtual magnetic monopole. The magnitude of the charge of such a virtual monopole is *continuously tunable*: it is given by a unit charge $g_0 = -\frac{1}{2}$ multiplied by the factor $\sqrt{\gamma^2-\rho^2}/\gamma \in (0,1]$ for $\gamma > 0$ or $\sqrt{\gamma^2-\rho^2}/\gamma \in [-1,0)$ for $\gamma < 0$. The singular $\boldsymbol{n}^z$ component of (5.19) is analogous to the so-called Dirac string. In the $\mathcal{PT}$-symmetric quantum system studied here, it generates a flux in the range of $[0,\pi)$ or $(\pi,2\pi]$; therefore, this string is measurable. Note that the results in (5.19) are based on the choice of $K = K^\dagger$. The unitary time evolution does not uniquely specify the operator K. One may choose a different K and may get different results in (5.19). For example, in [16] a different evolution equation was proposed and different results were obtained.

6. Conclusions

In this paper, we give explicit parametrizations of general 2×2 $\mathcal{PT}$-symmetric and $\tilde{\mathcal{P}}$-pseudo-Hermitian matrix Hamiltonians and the associated metric operators. We also show the equivalence of these two formulations in two dimensions. We review the application of these matrices in time-dependent quantum mechanics. For a $\mathcal{PT}$-symmetric two-level system, a Berry-like phase is found to display unusual features when compared with the conventional Hermitian magnetic monopole and Dirac string. There are many other applications to study and more unexpected results in $\mathcal{PT}$-symmetric quantum mechanics to explore.

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