Two- and four-dimensional representations of the $\mathcal{PT}$- and $\mathcal{CPT}$-symmetric fermionic algebras

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Fermionic systems differ from their bosonic counterparts, the main difference with regard to symmetry considerations being that $\mathcal{T}^2 = -1$ for fermionic systems. In $\mathcal{PT}$-symmetric quantum mechanics an operator has both $\mathcal{PT}$ and $\mathcal{CPT}$ adjoints. Fermionic operators η, which are quadratically nilpotent ($\eta^2 = 0$), and algebras with $\mathcal{PT}$ and $\mathcal{CPT}$ adjoints can be constructed. These algebras obey different anticommutation relations: $\eta\eta^{\mathcal{PT}} + \eta^{\mathcal{PT}}\eta = -\mathbb{1}$, where $\eta^{\mathcal{PT}}$ is the $\mathcal{PT}$ adjoint of η, and $\eta\eta^{\mathcal{CPT}} + \eta^{\mathcal{CPT}}\eta = \mathbb{1}$, where $\eta^{\mathcal{CPT}}$ is the $\mathcal{CPT}$ adjoint of η. This paper presents matrix representations for the operator η and its $\mathcal{PT}$ and $\mathcal{CPT}$ adjoints in two and four dimensions. A $\mathcal{PT}$-symmetric second-quantized Hamiltonian modeled on quantum electrodynamics that describes a system of interacting fermions and bosons is constructed within this framework and is solved exactly.

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I. INTRODUCTION

A complex Hamiltonian that is $\mathcal{PT}$ symmetric (invariant under space-time reflection) may exhibit two phases separated by a phase-transition point: an unbroken-$\mathcal{PT}$-symmetric phase in which the energy spectrum is entirely real and a broken-$\mathcal{PT}$-symmetric phase in which the spectrum is partly real and partly complex [1]. Complex Hamiltonians have been studied extensively in quantum mechanics and in quantum field theory. Most of this work has been devoted to the study of *bosonic* theories, for which $\mathcal{T}^2 = 1$. However, $\mathcal{T}^2 = -1$ for fermionic theories, a crucial feature that leads to differences in the formalism. For example, if the Hamiltonian H has a real eigenvalue, then H has a corresponding degenerate pair of eigenvectors, ϕ and $\mathcal{PT}\phi$; this is a consequence of Kramer's theorem for ordinary quantum mechanics. Non-Hermitian fermionic systems have been studied within the wider framework of pseudo-Hermiticity [2].

A previous paper [3] investigated a matrix representation of a nilpotent fermionic operator η satisfying $\eta^2 = 0$ together with an adjoint nilpotent operator, denoted generically by $\bar{\eta}$. These operators satisfied a fermionic anticommutator relation $\eta\bar{\eta} + \bar{\eta}\eta = \epsilon\mathbb{1}$. The value $\epsilon = 0$ corresponds to a Grassmann algebra and the value $\epsilon = 1$ corresponds to a standard fermionic operator anticommutation relation. However, the value $\epsilon = -1$ was obtained for this anticommutation relation in a specific case of a four-dimensional matrix. Subsequently, Cherbal and Trifonov formalized this result [4], making use of the non-Hermitian formulation of quantum mechanics in Ref. [2] and the notation of Ref. [5].

The problem with determining the value of ϵ for the anticommutator lies in the definition of the adjoint nilpotent element $\bar{\eta}$. In Ref. [3] $\bar{\eta}$ was chosen to be the $\mathcal{PT}$ *reflection* of η; that is,

$$\bar{\eta} = \mathcal{PT}\eta\mathcal{T}^{-1}\mathcal{P}^{-1}.$$

This paper revises the definition of $\bar{\eta}$ in order to make it consistent with the concept of a fermionic inner product. With this revision, the fermionic algebra using $\eta^{\mathcal{PT}}$, the $\mathcal{PT}$ adjoint of η, always gives rise to an anticommutation relation with $\epsilon = -1$. However, if we use $\eta^{\mathcal{CPT}}$, the $\mathcal{CPT}$ adjoint, the fermionic algebra becomes the conventional Hermitian fermionic algebra $\epsilon = 1$.

Knowing the structural properties of the fermionic operators is a technical but important issue as it provides the basis for constructing theories of many-body systems in second quantization. It is particularly useful in the context of a given symmetry, such as $\mathcal{PT}$ symmetry, because the Hamilton or Lagrange functions constructed in this way automatically have the symmetry properties required. The second-quantized approach enables one to describe and analyze dynamic systems. We illustrate this formalism with an exactly solvable model of a $\mathcal{PT}$-symmetric Hamiltonian for fermions interacting with bosons. This model is based on the structure of quantum electrodynamics. We solve this Hamiltonian exactly for the eigenvalues and calculate the renormalized mass of the fermion.

This paper is organized as follows. In Sec. II we review the choice of the inner product in order to set our notation and we define the $\mathcal{PT}$ and $\mathcal{CPT}$ adjoints using appropriate definitions of the $\mathcal{PT}$ and $\mathcal{CPT}$ inner products. In Secs. III and IV, we investigate two- and four-dimensional operator algebras and seek a general ansatz for the matrix representation of η and its respective $\mathcal{PT}$ and $\mathcal{CPT}$ adjoints, which we denote as $\eta^{\mathcal{PT}}$ and $\eta^{\mathcal{CPT}}$. In Sec. V we present our calculation of a simple model of a $\mathcal{PT}$-symmetric Hamiltonian of fermions interacting with bosons. Concluding remarks are made in Sec. VI.

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II. $\mathcal{PT}$ AND $\mathcal{CPT}$ ADJOINTS OF FERMIONIC OPERATORS

In this section we follow the approach of Ref. [5] and describe the general (abstract) formulation. As the parity operator $\mathcal{P}$ is linear, its action on the wave function of a finite-dimensional system can be expressed by a matrix S, $\mathcal{P}\psi = S\psi$. Since parity applied twice yields the identity matrix, it follows that $S^2 = 1$ and that the matrix S must have the eigenvalues ± 1. In contrast, the time-reversal operator is antilinear, so its action on the system can be expressed by a matrix Z combined with the complex-conjugate operation on the function it operates on, $\mathcal{T}\psi = Z\psi^*$. It is assumed that $[\mathcal{P},\mathcal{T}] = 0$. In terms of these symbols, the $\mathcal{PT}$ inner product for fermions is defined as

$$(\phi,\psi)_{\mathcal{PT}} \equiv (\mathcal{PT}\phi)^T Z\psi. \tag{1}$$

Thus, the $\mathcal{PT}$ adjoint of any operator A is defined by

$$(A^{\mathcal{PT}}\phi,\psi)_{\mathcal{PT}} \equiv (\phi, A\psi)_{\mathcal{PT}}. \tag{2}$$

As was done in Ref. [5], we insert the definition (1) into the left and right sides of Eq. (2), set $A = \eta$, and extract the operator relation

$$\eta^{\mathcal{PT}} = S\eta^\dagger S. \tag{3}$$

This is the $\mathcal{PT}$ adjoint for fermionic systems.

Let us examine the anticommutator of η with $\eta^{\mathcal{PT}}$. According to [2], we obtain a fermionic algebra with a minus sign:

$$\eta\eta^{\mathcal{PT}} + \eta^{\mathcal{PT}}\eta = -\mathbb{1}. \tag{4}$$

The minus sign is a signal that the $\mathcal{PT}$ inner product is not positive definite.

Following [1], one needs to introduce an additional operator $\mathcal{C}$ in order to change the $-$ sign in Eq. (4) to a $+$ sign. This operator thus reflects the sign of the norm.[1] The operator $\mathcal{C}$ is linear; it is thus represented by a matrix K. Then the $\mathcal{CPT}$ inner product is defined as [5]

$$(\phi,\psi)_{\mathcal{CPT}} = (\mathcal{CPT}\phi)^T Z\psi = (KSZ\phi^*)^T Z\psi,$$

and, after some algebra, this takes the form

$$(\phi,\psi)_{\mathcal{CPT}} = \phi^\dagger SK\psi.$$

As a consequence, $A^{\mathcal{CPT}}$, the $\mathcal{CPT}$ adjoint of an operator A, is defined by

$$(A^{\mathcal{CPT}}\phi,\psi)_{\mathcal{CPT}} = (\phi, A\psi)_{\mathcal{CPT}},$$

and thus $A^{\mathcal{CPT}}$ is given by the operator relation

$$A^{\mathcal{CPT}} = KSA^\dagger SK.$$

The $\mathcal{CPT}$ adjoint is related to the $\mathcal{PT}$ adjoint by

$$A^{\mathcal{CPT}} = KA^{\mathcal{PT}}K.$$

[1]The mathematical properties of the $\mathcal{C}$ operator resemble those of the charge-conjugation operator of Dirac, but in this context $\mathcal{C}$ plays a completely different role, simply forcing the norm of the state vectors to be positive.

In accordance with [2], the anticommutator of a fermionic operator η with its $\mathcal{CPT}$ adjoint should satisfy a *conventional* fermionic algebra

$$\eta\eta^{\mathcal{CPT}} + \eta^{\mathcal{CPT}}\eta = \mathbb{1}. \tag{5}$$

III. TWO-DIMENSIONAL η, $\eta^{\mathcal{PT}}$, $\eta^{\mathcal{CPT}}$

A. Real representations of η and $\eta^{\mathcal{PT}}$

We seek a two-dimensional matrix representation in which $\eta^{\mathcal{PT}}$ is the $\mathcal{PT}$ adjoint of η in accordance with Eq. (3). A general matrix

$$\eta = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \tag{6}$$

whose square vanishes, has a vanishing trace and determinant. Let us assume that a, b, and c are real numbers. The parameter a is fixed by the determinant condition

$$a^2 + bc = 0. \tag{7}$$

In two dimensions parity reflection $\mathcal{P}$ can be represented by σ_x, a real symmetric matrix whose square is unity:

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We then find that

$$\eta^{\mathcal{PT}} = \begin{pmatrix} -a & b \\ c & a \end{pmatrix}, \tag{8}$$

which satisfies the nilpotency condition $(\eta^{\mathcal{PT}})^2 = 0$. Now, evaluating the anticommutator of η with $\eta^{\mathcal{PT}}$, we find that

$$\eta\eta^{\mathcal{PT}} + \eta^{\mathcal{PT}}\eta = \mathrm{diag}(-4a^2). \tag{9}$$

For nonvanishing values of a the anticommutator (9) is negative and with the choice $a^2 = 1/4$ it can be normalized to $\eta\eta^{\mathcal{PT}} + \eta^{\mathcal{PT}}\eta = -\mathbb{1}$.

Because the right side of Eq. (9) is nonpositive the standard fermionic algebra with $\epsilon = +1$ does not have a 2×2 representation. But the right side of Eq. (9) can vanish if we take $a = 0$. Thus, the Grassmann algebra has a nontrivial representation. For example, we may take

$$\eta = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}.$$

This result differs from the conventional Hermitian fermionic algebra, where the standard algebra has a representation but the Grassmann algebra does not [3].

B. $\mathcal{CPT}$ adjoint

We have not specified the Hamiltonian, which is required to calculate the $\mathcal{CPT}$ product. Nevertheless, we can still determine the $\mathcal{CPT}$ transformed operator $\eta^{\mathcal{CPT}}$ that yields the standard fermionic algebra (5). To do so, we first use the fact that $\mathcal{C}$ and $\mathcal{PT}$ commute to obtain a general form for the matrix K

$$K = \begin{pmatrix} g & B \\ A & -g \end{pmatrix}, \tag{10}$$

where g, A, and B are arbitrary real parameters. Since $K^2 = \mathbb{1}$, we obtain the constraint $g^2 + AB = 1$. Hence, the $\mathcal{CPT}$ adjoint of η is

$$\eta^{\mathcal{CPT}} = \begin{pmatrix} -ag^2 + bgA + cgB + aAB & -2agB + cB^2 - bg^2 \\ -2agA + bA^2 - cg^2 & ag^2 - bgA - cgB - aAB \end{pmatrix}.$$

The anticommutator of η and $\eta^{\mathcal{CPT}}$ is then

$$\eta\eta^{\mathcal{CPT}} + \eta^{\mathcal{CPT}}\eta = \mathrm{diag}(2a^2AB + c^2B^2 + b^2A^2).$$

By using the determinant relation (7) we eliminate a^2 and find that $(bA - cB)^2 = 1$, which links the parameters A,B to b,c. The choice $bA = cB$ seems to yield the Grassmann algebra $\eta\eta^{\mathcal{CPT}} + \eta^{\mathcal{CPT}}\eta = 0$. However, we shall see in Sec. III D that because $\mathcal{C}$ and the Hamiltonian commute, the choice $bA = cB$ is ruled out, and we arrive at the same result as in the conventional Hermitian fermionic algebra.

C. Ground state, excited state, and number operator

We can normalize the anticommutator $\eta\eta^{\mathcal{PT}} + \eta^{\mathcal{PT}}\eta$ to $-\mathbb{1}$ by rescaling η and $\eta^{\mathcal{PT}}$ by $2a$. In this case Eqs. (6) and (8) become

$$\eta = \frac{1}{2a}\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad \eta^{\mathcal{PT}} = \frac{1}{2a}\begin{pmatrix} -a & b \\ c & a \end{pmatrix}.$$

We then define the ground state $|0\rangle$ as that state that is annihilated by η: $\eta|0\rangle = 0$. Using Eq. (6), we represent this state as

$$|0\rangle = \begin{pmatrix} -b \\ a \end{pmatrix}.$$

To create the $\mathcal{PT}$-symmetric state $|1\rangle$ we operate on $|0\rangle$ with $\eta^{\mathcal{PT}}$ and get

$$|1\rangle = \begin{pmatrix} b \\ a \end{pmatrix}.$$

We define the $\mathcal{PT}$ number operator as

$$N^{\mathcal{PT}} = \eta^{\mathcal{PT}}\eta$$

and establish by direct calculation that

$$N^{\mathcal{PT}}|0\rangle = 0, \qquad N^{\mathcal{PT}}|1\rangle = -|1\rangle.$$

Evidently, $N^{\mathcal{PT}}$ gives the *negative* of the state occupation number. We use this fact in Sec. V in constructing a second-quantized form of a $\mathcal{PT}$-symmetric fermionic Hamiltonian.

D. General two-dimensional $\mathcal{PT}$-symmetric Hamiltonian

A consistent fermionic $\mathcal{PT}$ quantum mechanics must satisfy three conditions: (i) The Hamiltonian must be self-adjoint with respect to the $\mathcal{PT}$ inner product for fermions; that is, the definition (1) must hold; (ii) H must commute with $\mathcal{PT}$; and (iii) the $\mathcal{PT}$ symmetry must be unbroken. The first two criteria give the following general form for a real Hamiltonian:

$$H = \begin{pmatrix} \alpha & \beta \\ \gamma & \alpha \end{pmatrix} \qquad (\alpha, \beta, \gamma \ \text{ real}). \tag{11}$$

The matrix representations of the parity and time-reversal operators, that is, S and Z in Eq. (1), are given by σ_x.

In Sec. III B we obtained the matrix representation (10) associated with the $\mathcal{C}$ operator. A property of $\mathcal{C}$ not considered in Sec. III B is that $\mathcal{C}$ commutes with H. The commutation of K and H forces g to vanish, so the earlier constraint $g^2 + AB = 1$ reduces to $AB = 1$.

Sec. III B concludes that if $bA = cB$, one obtains a representation for a Grassmann algebra. However, the determinant condition $a^2 + bc = 0$ implies that bc is a nonpositive quantity. Therefore, to have $bA = cB$, AB must also be nonpositive, which contradicts the constraint $AB = 1$. Thus, as in the conventional Hermitian case, the Grassmann algebra does not have a nontrivial representation.

The eigenvalues of Eq. (11) are

$$\lambda_\pm = \alpha \pm \sqrt{\beta\gamma}, \tag{12}$$

and the corresponding eigenvectors are

$$|\lambda_+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt[4]{\frac{\beta}{\gamma}} \\ \sqrt[4]{\frac{\gamma}{\beta}} \end{pmatrix}, \quad |\lambda_-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt[4]{\frac{\beta}{\gamma}} \\ -\sqrt[4]{\frac{\gamma}{\beta}} \end{pmatrix}.$$

The formula (12) indicates that if β and γ are positive, the symmetry is unbroken; that is, the eigenvalues are real.

It is easy to establish that

$$\langle\lambda_+|\lambda_+\rangle_{\mathcal{PT}} = 1, \qquad \langle\lambda_-|\lambda_-\rangle_{\mathcal{PT}} = -1,$$
$$\langle\lambda_+|\lambda_-\rangle_{\mathcal{PT}} = \langle\lambda_-|\lambda_+\rangle_{\mathcal{PT}} = 0.$$

We introduce $\mathcal{C}$ as a measure of the sign of the norm:

$$\mathcal{C}|\lambda_+\rangle = |\lambda_+\rangle, \qquad \mathcal{C}|\lambda_-\rangle = -|\lambda_-\rangle.$$

The matrix representation of $\mathcal{C}$ is then

$$K = \begin{pmatrix} 0 & \sqrt{\beta/\gamma} \\ \sqrt{\gamma/\beta} & 0 \end{pmatrix}.$$

For the Hamiltonian (11), the annihilation operator now reads

$$\eta = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{\beta/\gamma} \\ -\sqrt{\gamma/\beta} & -1 \end{pmatrix}.$$

As expected, η is nilpotent and

$$\eta|\lambda_-\rangle = 0, \qquad \eta|\lambda_+\rangle = |\lambda_-\rangle.$$

We now obtain the $\mathcal{PT}$ adjoint of η as

$$\eta^{\mathcal{PT}} = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{\beta/\gamma} \\ -\sqrt{\gamma/\beta} & 1 \end{pmatrix}.$$

Defining the $\mathcal{PT}$ number operator to be $N^{\mathcal{PT}} = \eta^{\mathcal{PT}}\eta$, we can show that

$$\{N,\eta\}_+ = -\eta, \qquad \{N,\eta^{\mathcal{PT}}\}_+ = -\eta^{\mathcal{PT}}.$$

The minus sign implies that the $\mathcal{PT}$ number operator $N^{\mathcal{PT}}$ gives the negative of the state occupation number, as discussed in Sec. III C.

In addition, we remark that the Hamiltonian of our $\mathcal{PT}$-symmetric interacting fermions can be recast as a free bosonic Hamiltonian:

$$H = \Delta\lambda(-N^{\mathcal{PT}}) + \lambda_{-}\mathbb{1},$$

where $\Delta\lambda = \lambda_{+} - \lambda_{-}$.

The anticommutator $\eta\eta^{\mathcal{PT}} + \eta^{\mathcal{PT}}\eta = -\mathbb{1}$, but if instead we use the $\mathcal{CPT}$ adjoint of η,

$$\eta^{\mathcal{CPT}} = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{\beta/\gamma} \\ \sqrt{\gamma/\beta} & -1 \end{pmatrix},$$

we obtain the conventional anticommutator $\eta\eta^{\mathcal{CPT}} + \eta^{\mathcal{CPT}}\eta = \mathbb{1}$.

IV. FOUR-DIMENSIONAL η, $\eta^{\mathcal{PT}}$, $\eta^{\mathcal{CPT}}$

A general set of 12-parameter complex nilpotent matrices was proposed in Ref. [3] as

$$\eta = \begin{pmatrix} -ch-bg-af & f & g & h \\ -a(ch+bg+af) & af & ag & ah \\ -b(ch+bg+af) & bf & bg & bh \\ -c(ch+bg+af) & cf & cg & ch \end{pmatrix}, \tag{13}$$

where a, b, c, f, g, and h are arbitrary complex numbers. This form was constructed assuming that the trace of η as well as its determinant must vanish in order to guarantee nilpotency. We use the convention of Ref. [5] for the matrix representations of S and Z; that is,

$$S = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad Z = \begin{pmatrix} e_2 & 0 \\ 0 & e_2 \end{pmatrix}, \tag{14}$$

where I is the 2×2 identity matrix, and e_2 is $e_2 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The $\mathcal{PT}$ adjoint of η reads

$$\eta^{\mathcal{PT}} = \begin{pmatrix} -F^* & -a^*F^* & b^*F^* & c^*F^* \\ f^* & a^*f^* & -b^*f^* & -c^*f^* \\ -g^* & -a^*g^* & b^*g^* & c^*g^* \\ -h^* & -a^*h^* & b^*h^* & c^*h^* \end{pmatrix},$$

where $F = ch + bg + af$. As required, $\eta^{\mathcal{PT}}$ is also nilpotent. One can evaluate the anticommutator of η and $\eta^{\mathcal{PT}}$. This is found to be

$$\eta\eta^{\mathcal{PT}} + \eta^{\mathcal{PT}}\eta = \begin{pmatrix} J+|F|^2K & a^*J - F^*fK & -b^*J - F^*gK & -c^*J - F^*hK \\ aJ - f^*FK & |a|^2J + |f|^2K & -ab^*J + f^*gK & -ac^*J + f^*hK \\ bJ + g^*FK & ba^*J - g^*fK & -|b|^2J - |g|^2K & -bc^*J - g^*hK \\ cJ + h^*FK & ca^*J - h^*fK & -cb^*J - h^*gK & -|c|^2J - |h|^2K \end{pmatrix},$$

where $J = |F|^2 + |f|^2 - |g|^2 - |h|^2$ and $K = 1 + |a|^2 - |b|^2 - |c|^2$.

To obtain the fermionic algebras, it is necessary that the off-diagonal terms vanish. This gives the relations

$$a^*F = -f, \quad b^*F = g, \quad c^*F = h. \tag{15}$$

However, these relations force the diagonal terms to vanish. The particular choice of η in Eq. (13) proposed in Ref. [3] is only suitable for constructing a $\mathcal{PT}$-symmetric Grassmann algebra, where the anticommutator $\{\eta,\eta^{\mathcal{PT}}\}$ vanishes. An example of an η that satisfies the relations (15) and leads to a Grassmann algebra is

$$\eta = \begin{pmatrix} 1 & 1 & i & -i \\ 1 & 1 & i & -i \\ i & i & -1 & 1 \\ -i & -i & 1 & -1 \end{pmatrix}.$$

Let us examine another set of matrices that cannot be obtained from Eq. (13):

$$\eta = \begin{pmatrix} f & 0 & \alpha c & \alpha b \\ 0 & f & \alpha b^* & -\alpha c^* \\ \beta c^* & \beta b & -f & 0 \\ \beta b^* & -\beta c & 0 & -f \end{pmatrix}, \tag{16}$$

where b and c are complex and f, α, and β are real arbitrary parameters. This ansatz is a block-form construct with 2×2 matrices that ensures that the matrix is traceless in the simplest possible fashion. In addition, the off-diagonal elements have been chosen to be scaled Hermitian conjugates of one another, introducing a minimum number of parameters. Nilpotency of η must now be enforced and leads to the requirement that

$$f^2 + \alpha\beta(|b|^2 + |c|^2) = 0. \tag{17}$$

Using the matrix representations of S and Z in Eq. (14), we obtain the $\mathcal{PT}$ adjoint of η:

$$\eta^{\mathcal{PT}} = \begin{pmatrix} f & 0 & -\beta c & -\beta b \\ 0 & f & -\beta b^* & \beta c^* \\ -\alpha c^* & -\alpha b & -f & 0 \\ -\alpha b^* & \alpha c & 0 & -f \end{pmatrix}.$$

Equation (17) implies that $\eta^{\mathcal{PT}}$ is also nilpotent.

The anticommutator of η and $\eta^{\mathcal{PT}}$ is

$$\eta\eta^{\mathcal{PT}} + \eta^{\mathcal{PT}}\eta = \mathrm{diag}\{2f^2 - (\alpha^2 + \beta^2)(|b|^2 + |c|^2)\},$$

and because of Eq. (17) this reduces to

$$\eta\eta^{\mathcal{PT}} + \eta^{\mathcal{PT}}\eta = -\mathrm{diag}\{(\alpha + \beta)^2(|b|^2 + |c|^2)\}.$$

Thus, the anticommutator is nonpositive. The choice $\alpha = -\beta$ gives rise to a nontrivial representation for the Grassmann algebra. However, when $\alpha \neq -\beta$, the above anticommutator with suitable normalization can be written as

$$\eta\eta^{\mathcal{PT}} + \eta^{\mathcal{PT}}\eta = -\mathbb{1}.$$

To obtain the standard fermionic algebra we again consider the $\mathcal{CPT}$ adjoint of η instead of $\eta^{\mathcal{PT}}$. We construct the $\mathcal{C}$ operator as follows. We note that the commutation of $\mathcal{C}$ and $\mathcal{PT}$ gives

$$KSZ = SZK^*, \tag{18}$$

where K, S, and Z are the matrix representations of $\mathcal{C}$, $\mathcal{P}$, and $\mathcal{T}$. Another characteristic of the $\mathcal{C}$ operator is that it commutes with the Hamiltonian. The procedure to construct a general $\mathcal{PT}$-symmetric Hamiltonian for fermionic systems is described in Ref. [5]. A matrix K that satisfies the two criteria in Eq. (18) and $[\mathcal{C},H]=0$ is parametrized as

$$K=\begin{pmatrix} g & 0 & -\gamma c & -\gamma b\\ 0 & g & -\gamma b^* & \gamma c^*\\ \gamma c^* & \gamma b & -g & 0\\ \gamma b^* & -\gamma c & 0 & -g\end{pmatrix},$$

where g and γ are real numbers.

The requirement $K^2=\mathbb{1}$ leads to the additional constraint

$$g^2-\gamma^2(|b|^2+|c|^2)=1. \tag{19}$$

Having found K, we can easily obtain the $\mathcal{CPT}$ adjoint of η:

$$\eta^{\mathcal{CPT}}=\begin{pmatrix} D & 0 & -cA & -bA\\ 0 & D & -b^*A & c^*A\\ c^*B & bB & -D & 0\\ b^*B & -cB & 0 & -D\end{pmatrix},$$

where

$$\begin{aligned} D&=fg^2+(|b|^2+|c|^2)\gamma(\gamma f+\alpha g-\beta g),\\ A&=2\gamma fg-\beta g^2+\alpha\gamma^2(|b|^2+|c|^2),\\ B&=2\gamma fg+\alpha g^2-\beta\gamma^2(|b|^2+|c|^2).\end{aligned}$$

Finally, the anticommutation of η and $\eta^{\mathcal{CPT}}$ reads

$$\begin{aligned}&\eta\eta^{\mathcal{CPT}}+\eta^{\mathcal{CPT}}\eta\\ &\quad=\mathrm{diag}\{(|b|^2+|c|^2)[2\gamma f+(\alpha-\beta)g]^2\},\end{aligned}$$

where Eqs. (17) and (19) have been used. Note that the anticommutator is *positive* and with a suitable normalization can be written as

$$\eta\eta^{\mathcal{CPT}}+\eta^{\mathcal{CPT}}\eta=\mathbb{1}.$$

For completeness, we remark that the ground state can be defined, as in Sec. III C, as being the state that is annihilated by η: $\eta|0\rangle=0$. Using Eq. (16), we represent this state as

$$|0\rangle=\begin{pmatrix} f\\ 0\\ \beta c^*\\ \beta b^*\end{pmatrix}.$$

To create the $\mathcal{PT}$-symmetric state $|1\rangle$ we operate on $|0\rangle$ with $\eta^{\mathcal{PT}}$ and obtain

$$|1\rangle=\begin{pmatrix} \beta(|b|^2+|c|^2)\\ 0\\ fc^*\\ fb^*\end{pmatrix}.$$

Following the procedure in Sec. III C, after normalizing $|0\rangle$ and $|1\rangle$ above, we ascertain by direct calculation that $N^{\mathcal{PT}}|0\rangle=0$ and $N^{\mathcal{PT}}|1\rangle=-|1\rangle$, where we have used $N^{\mathcal{PT}}=\eta^{\mathcal{PT}}\eta$ and Eq. (17), thus illustrating again that $N^{\mathcal{PT}}$ yields the *negative* of the state occupation number.

FIG. 1. The fermion (solid line) can emit or absorb bosons (wavy gray lines). These are the only possible interactions, so the fermion number is conserved.

V. SIMPLE MODEL HAMILTONIAN

In this section we construct a $\mathcal{PT}$-symmetric model of interacting fermions and bosons. The idea is based on the Lee model in which the lack of crossing symmetry makes the model exactly solvable [6]. We consider a single fermion that may emit and absorb bosons, as shown in Fig. 1, but the bosons may not produce a fermion-antifermion pair.

A Hamiltonian that describes this system is

$$H=ma^\dagger a-M\eta^{\mathcal{PT}}\eta-ga^\dagger\eta^{\mathcal{PT}}\eta-ga\eta^{\mathcal{PT}}\eta,$$

where the operator $a^\dagger$ creates (normal) bosons, but the fermionic operator $\eta^{\mathcal{PT}}$ creates a $\mathcal{PT}$-symmetric fermion. Here, m and M are the bare boson and fermion masses and g is the coupling amplitude. This Hamiltonian is not Hermitian but it is $\mathcal{PT}$ symmetric.

A state containing a single bare fermion and any number n of bare bosons can be written as $|E\rangle=\sum_{n=0}^{\infty}c_n|1,n\rangle$. We assume that this state is normalized; that is,

$$\langle E|E\rangle=\sum_{n=0}^{\infty}c_n^2<\infty. \tag{20}$$

The annihilation and creation operators for bosons obey $a|n\rangle=\sqrt{n}|n-1\rangle$ and $a^\dagger|n\rangle=\sqrt{n+1}|n+1\rangle$. In addition, $a^\dagger a$ is the boson number operator; that is, $a^\dagger a|n\rangle=n|n\rangle$. For the case of fermions we have the peculiar anticommutation relation $\eta\eta^{\mathcal{PT}}+\eta^{\mathcal{PT}}\eta=-\mathbb{1}$. However, as in Sec. III, we interpret η as a lowering operator and $\eta^{\mathcal{PT}}$ as a raising operator. Thus, the fermion number operator is $-\eta^{\mathcal{PT}}\eta$.

The time-independent Schrödinger equation $H|E\rangle=E|E\rangle$ takes the form

$$\begin{aligned}&\sum_{n=0}^{\infty}mnc_n|1,n\rangle+\sum_{n=0}^{\infty}Mc_n|1,n\rangle\\ &\quad+\sum_{n=0}^{\infty}g\sqrt{n+1}c_n|1,n+1\rangle\\ &\quad+\sum_{n=0}^{\infty}g\sqrt{n}c_n|1,n-1\rangle=\sum_{n=0}^{\infty}Ec_n|1,n\rangle.\end{aligned}$$

We shift indices and pick off the coefficients of $|1,n\rangle$ to obtain a recursion relation c_n:

$$(mn+M)c_n+g\sqrt{n}c_{n-1}+g\sqrt{n+1}c_{n+1}=Ec_n.$$

The substitution $c_n=d_n\sqrt{n!}$ gives the simpler recursion relation

$$(mn+M)d_n+gd_{n-1}+g(n+1)d_{n+1}=Ed_n. \tag{21}$$

For large n we can neglect the Md_n and Ed_n terms and obtain an approximate equation for d_n that is valid for large n:

$$mnd_n + gd_{n-1} + g(n+1)d_{n+1} \simeq 0.$$

There are two consistent asymptotic dominant balances for $n \gg 1$: If the first and second terms balance for large n, then

$$d_n \simeq (-g/m)^n/n!; \tag{22}$$

if the first and third terms balance, then

$$d_n \simeq (-m/g)^n. \tag{23}$$

(A dominant balance between the second and third terms is inconsistent.) The norm in Eq. (20) becomes $\sum_{n=0}^{\infty} d_n^2 n!$. Therefore, Eq. (22) is acceptable but Eq. (23) is not.

Next, we define a generating function $f(x) \equiv \sum_{n=0}^{\infty} d_n x^n$; if Eq. (22) holds, then $f(x)$ is an entire function of x, but if Eq. (23) holds, we see that $f(x)$ has a finite radius of convergence with a singularity in the complex-x plane at $x = -g/m$.

If we multiply Eq. (21) by x^n and sum from 0 to ∞, we obtain the first-order differential equation

$$(mx + g)f'(x) = (E - M - xg)f(x),$$

whose solution is

$$f(x) = Ke^{-gx/m}(mx + g)^{E/m - M/m + g^2/m^2}.$$

As predicted, there is a singularity at $x = -g/m$ unless the exponent in the second term on the right side is a non-negative integer $N = 0, 1, 2, \ldots$. This yields the *exact* spectrum of physical fermion states:

$$E_N = Nm + M - g^2/m \quad (N = 0, 1, 2, \ldots).$$

Note that as a consequence of the interaction, the mass $M - g^2/m$ of the physical fermion is *lower* than the mass M of the bare fermion.

VI. BRIEF CONCLUDING REMARKS

In this paper we have used the alternative formalism for the fermionic scalar product in Ref. [5] to reexamine the operator algebra for fermions in the context of $\mathcal{PT}$ symmetry. We have investigated general matrix representations of the $\mathcal{PT}$ and $\mathcal{CPT}$ fermionic creation and destruction operators without making direct reference to a Hamiltonian. Knowing the behavior of such operators, especially $\mathcal{PT}$ operators, can be important for many-body theory, which often uses the operator definitions to construct the Hamiltonian (in second-quantized form). It can also be important in understanding the nature of species oscillation in neutrinos [7].

We have examined the operator algebras in detail for 2×2 matrices and for the 4×4 case. Using the algebra that we have developed, we apply the peculiar anticommutation relations pertinent to the $\mathcal{PT}$ algebra to construct a second-quantized $\mathcal{PT}$-symmetric quantum field theory, namely, a solvable low-dimensional model of electrodynamics (a modified Lee model) for which the renormalized energy spectrum is calculated in a closed form and is found to be real.

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