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Geometric phase, bundle classification, and group representation

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The line bundles that arise in the holonomy interpretations of the geometric phase display curious similarities to those encountered in the statement of the Borel–Weil–Bott theorem of the representation theory. The remarkable relationship between the mathematical structure of the geometric phase and the classification theorem for complex line bundles provides the necessary tools for establishing the relevance of the Borel–Weil–Bott theorem to Berry's adiabatic phase. This enables one to define a set of topological charges for arbitrary compact connected semisimple dynamical Lie groups. These charges signify the topological content of the phase. They can be explicitly computed. In this paper, the problem of the determination of the parameter space of the Hamiltonian is also addressed. It is shown that, in general, the parameter space is either a flag manifold or one of its submanifolds. A simple topological argument is presented to indicate the relation between the Riemannian structure on the parameter space and Berry's connection. The results about the fiber bundles and group theory are used to introduce a procedure to reduce the problem of the nonadiabatic (geometric) phase to Berry's adiabatic phase for cranked Hamiltonians. Finally, the possible relevance of the topological charges of the geometric phase to those of the non-Abelian monopoles is pointed out. © *1996 American Institute of Physics.* [S0022-2488(96)03502-7]

I. INTRODUCTION

In the past ten years, since the revival of the geometric phase,[1,2] by Berry,[3] the subject has attracted the attention of many physicists. The main reason for the unusual popularity of this remarkably simple subject, particularly among the theoretical physicists, has been its rich mathematical and physical foundations.

Recently, it was shown that the two holonomy interpretations of Berry's phase were linked via the theory of universal bundles.[4,5] This remarkable coincidence of the physics of geometric phase and the mathematics of fiber bundles enables one to set up a convenient framework to analyze the nonadiabatic phase.[5] In the present paper, the results of[5] are briefly reviewed and their generalization to arbitrary finite-dimensional unitary systems are presented.

In Sec. II, it is shown how the study of the standard example of a spin in a processing magnetic field directs one to the Borel–Weil–Bott (BWB) theorem of the representation theory of compact semisimple Lie groups. In Sec. III, the relation of the BWB theorem to the phenomenon of a geometric phase is discussed in a general setting. Section IV is devoted to a discussion of the relation of Berry's connection and the Riemannian geometry of the parameter space. Section V includes the discussion of the reduction of the nonadiabatic phase problem to the adiabatic one for the cranked Hamiltonians. Section VI consists of a short account on the classification of the

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parameter spaces and the topology of non-Abelian monopoles. Section VII includes the conclusions.

II. BUNDLE CLASSIFICATION AND THE HOLONOMY INTERPRETATIONS OF THE GEOMETRIC PHASE

There are two mathematical interpretations of Berry's (adiabatic) phase. These are due to Simon[6] and Aharonov and Anandan.[7] I shall refer to these two approaches by "BS" and "AA," which are the abbreviations of "Berry–Simon" and "Aharonov–Anandan," respectively.

In the BS approach, one constructs a line bundle L over the space M of the parameters of the system. Then, L is endowed with a particular connection that reproduces Berry's phase as the holonomy of the closed loop in the parameter space.

Let us consider a quantum mechanical system whose evolution is governed by a parameter-dependent Hamiltonian:

$$H=H(x),\quad x\in M.$$

Assume that for all $x\in M$ the spectrum of $H(x)$ is discrete and that there are no level crossings. Then, locally one can choose a set of orthonormal basic eigenstate vectors $\{|n,x\rangle\}$. As functions of x, $|n,x\rangle$ are smooth and single valued. By definition, they satisfy

$$H(x)|n,x\rangle=E_n(x)|n,x\rangle, \tag{1}$$

where $E_n(x)$ are the corresponding energy eigenvalues. The Hamiltonian is made explicitly time dependent by interpreting time t as the parameter of a curve,

$$C:[0,T]\ni t\to x(t)\in M, \tag{2}$$

and setting

$$H(t):=H(x(t)),\quad t\in[0,T]. \tag{3}$$

Then, each closed curve C in M defines a periodic Hamiltonian with period T. I shall discuss only the evolution of nondegenerate cyclic states with period T.

Under the adiabatic approximation the initial eigenstates undergo cyclic evolutions.[3] If $|\psi_n(t)\rangle$ denotes the evolving state vector, i.e., the solution of the Schrödinger equation:

$$H(t)|\psi_n(t)\rangle=i\,\frac{d}{dt}\,|\psi_n(t)\rangle$$

$$|\psi_n(0)\rangle:=|n,x(0)\rangle, \tag{4}$$

then

$$|\psi_n(T)\rangle\langle\psi_n(T)|\simeq|\psi_n(0)\rangle\langle\psi_n(0)|. \tag{5}$$

After a cycle is completed, the state vector gains a phase factor that consists of a dynamical ($e^{i\omega}$) and a geometric ($e^{i\gamma}$) part,

$$|\psi_n(T)\rangle=e^{i(\omega+\gamma)}|\psi_n(0)\rangle, \tag{6}$$

where

$$\omega := -\int_0^T E_n(x(t))dt,$$

and

$$e^{i\gamma} := \exp \oint_C A, \tag{7}$$

$$A := -\langle n,x|d|n,x\rangle = -\langle n,x| \frac{\partial}{\partial x^\mu} |n,x\rangle dx^\mu. \tag{8}$$

The one-form A is known as Berry's connection one-form.[3]

In Ref. 6, Simon showed that A could be interpreted as a connection one-form on a (spectral) line bundle L over M,

$$\mathbb{C} \to L \to M, \tag{9}$$

whose fibers are given by the energy eigenrays in the Hilbert space $\mathscr{H}$,

$$L_x := \{z|n,x\rangle \ : \ z \in \mathbb{C}\}. \tag{10}$$

Thus, in the BS approach, Berry's phase is identified with the holonomy of the loop $C \subset M$ defined by the connection one-form A of Eq. (8).

In the AA approach one considers a complex line bundle E, or alternatively, the associated $U(1)$-principal bundle, over the projective Hilbert space $P(\mathscr{H}) = \mathbb{C}P^N$, $N := \dim(\mathscr{H}) - 1$:

$$\mathbb{C} \to E \to \mathscr{P}(\mathscr{H}). \tag{11}$$

The fibers over the points $\eta = |\eta\rangle\langle\eta|$ of $\mathscr{P}(\mathscr{H}) = \mathbb{C}P^N$ are the corresponding rays:

$$E_\eta := \{z|\eta\rangle \ : \ z \in \mathbb{C}\}, \tag{12}$$

in the Hilbert space $\mathscr{H}$. (The topological structure of E is determined by the topological structure of $\mathbb{C}P^N$. In particular, a natural local trivialization is given by adopting the standard homogeneous local coordinate charts for $\mathbb{C}P^N$. The associated transition functions of E are determined from those of $\mathbb{C}P^N$ similarly. See Sec. IV for an alternative characterization of the topology of E.)

The AA connection one-form $\mathscr{A}$ (Ref. 7) is then viewed as a connection one-form on E and the geometric phase is identified with the corresponding holonomy of loops,

$$\mathscr{C}: [0,T] \ni t \to \eta(t) \in \mathscr{P}(\mathscr{H}), \tag{13}$$

in $\mathscr{P}(\mathscr{H})$. In the adiabatic approximation one approximates $\eta(t)$ by $\psi_n(t)$ of Eq. (4).

These two interpretations of Berry's phase turn out to be linked via the theory of *universal bundles*. It is shown in Refs. 4 and 5 that E (with $N \to \infty$) is indeed the universal classifying line bundle,[8–10] and as a result of the classification theorem for complex line bundles,[9,8,11] every complex line bundle can be obtained as a pullback bundle from E. In particular, there is a smooth map,

$$f: M \to \mathscr{P}(\mathscr{H}), \tag{14}$$

such that

$$L = f^*(E). \tag{15}$$

The map f is simply given by

$$f(x):=|n,x\rangle\langle n,x|. \tag{16}$$

Furthermore, the fact that the phase is obtained from either of A or $\mathscr{A}$ is a consequence of the theory of *universal connections*.[12,13] In fact, the AA connection $\mathscr{A}$ is precisely the universal connection, which yields all connections on all complex line bundles as pullback connections. In particular, Berry's connection on L is given by

$$A=f^*(\mathscr{A}). \tag{17}$$

These results are exploited in Ref. 5 to explore the quantum dynamics of Berry's original example:

$$H(x)=b\mathbf{x}\cdot\mathbf{J}, \quad \mathbf{x}\in S^2\subset\mathbb{R}^3, \tag{18}$$

where b is the Larmor frequency, $\mathbf{x}$ is the direction of the magnetic field, and $\mathbf{J}=(J_i)$, $i=1,2,3$, are the generators of rotations, $J_i\in so(3)=su(2)$. In Ref. 5, it is shown that if one considers the case of precessing magnetic field, i.e., precessing $\mathbf{x}$ about a fixed axis, then one can promote Simon's construction to the nonadiabatic case, namely, define a nonadiabatic analog of Berry's connection and identify the nonadiabatic phase with its holonomy. This can be done in general unless the frequency of precession, ω, becomes equal to b. In the northern hemisphere the nonadiabatic connection $\tilde{A}$ is given by

$$\tilde{A}=ik(1-\cos\tilde{\theta})d\phi, \tag{19}$$

where k labels an eigenvalue of $H(x)$ (alternatively an eigenvalue of J_3), and

$$\cos\tilde{\theta}:=\frac{\cos\theta-\nu}{\sqrt{\nu^2-2\nu\cos\theta+1}}, \tag{20}$$

$$\nu:=\frac{\omega}{b}. \tag{21}$$

Here (θ,ϕ) are the spherical coordinates $(\theta\in[0,\pi))$, and ν is the "slowness parameter".[14] The adiabatic limit is characterized by $\nu\to 0$. In this limit $\tilde{A}$ approaches to Berry's connection,

$$A=ik(1-\cos\theta)d\phi. \tag{22}$$

Note that unlike the adiabatic case ($\nu\to 0$), the cyclic states in the more general nonadiabatic case cannot be approximated by the eigenstates of the initial Hamiltonian. They are given as the eigenstates of the unitary time evolution operator at time T. This operator does not generally commute with the initial Hamiltonian, and they do not share simultaneous eigenstates.

The topology of a line bundle on S^2 is determined by its first Chern number,

$$c_1:=\frac{i}{2\pi}\int_{S^2}\Omega, \tag{23}$$

where Ω is the curvature two-form. For line bundles, the curvature two-form is obtained from the connection one-form by taking its ordinary exterior derivative.[15] A simple calculation shows that taking $\Omega=d\tilde{A}$ results in

$$c_1=-2k, \quad \text{for } \nu<1. \tag{24}$$

This is quite remarkable since the fact that c_1 is an integer agrees with the fact that k is a half-integer. The first statement is an algebraic topological result, whereas the second is related to group theory. One of the best known mathematical results that links these two disciplines is the celebrated Borel–Weil–Bott (BWB) theorem.[16–19]

Equation (24) may also be viewed as an example of a topological quantization of angular momentum. In the language of magnetic monopoles, which are relevant to the adiabatic case, $k=-c_1/2$ corresponds to the product of the electric and magnetic charges.[20,21]

III. BOREL–WEIL–BOTT THEOREM AND THE BERRY–SIMON LINE BUNDLES

The BWB theorem constructs all the finite-dimensional irreducible representations (irreps.) of semisimple compact Lie groups from the irreps. of their maximal tori. The construction is as follows.

Let G be a semisimple compact Lie group and T be a maximal torus. Let $\mathscr{G}$ and Υ be the Lie algebras of G and T, respectively. G can be viewed as a principal bundle over the quotient space G/T:[22]

$$T\rightarrow G\rightarrow G/T. \tag{25}$$

The homogeneous space G/T can be shown to have a canonical complex structure.[17] Since T is Abelian, its irreps. are one dimensional.[22] Thus, each irrep. Λ of T defines an associated complex line bundle L_Λ to (25):

$$\mathbb{C}\rightarrow L_\Lambda\rightarrow G/T. \tag{26}$$

Now, consider a Λ whose corresponding line bundle L_Λ is an ample (positive) line bundle. Then L_Λ has the structure of a holomorphic line bundle. BWB theorem asserts that all the irreps. of G are realized on the spaces of holomorphic sections of ample (positive) line bundles, L_Λ. In particular, the space $\mathscr{H}_\Lambda$ of the holomorphic sections of L_Λ provides the irrep. of G with maximal weight Λ.[18,17,19]

The simplest nontrivial example of the application of the BWB theorem is for $G=\mathrm{SU}(2)$. In this case, $T=U(1)=S^1$ and $G/T=S^2=\mathbb{C}P^1$. The bundle (25) is the Hopf bundle:[22]

$$U(1)=S^1\rightarrow SU(2)=S^3\rightarrow S^2. \tag{27}$$

Λ takes non-negative half-integers. It is usually denoted by j in QM. It is common knowledge that $j=0, \frac{1}{2},1,...,$ yield all the irreps. of SU(2) and that the j representation has dimension $2j+1$. The dimension of the space $\mathscr{H}_\Lambda$ can be given by an index theorem.[18,16] For SU(2), it is obtained by the Riemann–Roch theorem in the context of the theory of Riemann surfaces. The result is

$$\dim(\mathscr{H}_\Lambda)=c(L_\Lambda)=1+c_1(L_\Lambda), \tag{28}$$

where c and c_1 denote the total and first Chern numbers of L_Λ. This means that one must have

$$c_1(L_\Lambda)=2j. \tag{29}$$

Combining (24) and (29), one recovers the line bundle L_Λ as Simon's line bundle L of (9) for $k=-j$.

In the rest of this section, I shall try to show that there is a general relationship between the constructions used in the BWB theorem and those encountered in BS interpretation of Berry's phase. To proceed in this direction, let us consider the generalization of (18) to an arbitrary compact semisimple Lie group, namely, consider

$$H(x)=\epsilon\sum_{i=1}^{d} x^i J_i, \quad (x^i)\in\mathbb{R}^d-\{0\}. \tag{30}$$

Here J_i are the generators of G and ϵ is a constant with the dimension of energy. Since $H(x)$ is assumed to be Hermitian, J_i must be represented by Hermitian matrices. In other words, the group G is in a unitary representation. In this sense, the example of $G=U(N)$ plays a universal role. (This reminds one of the Peter–Weyl theorem.[19,22])

The system described by Eq. (30) is studied in Refs. 23 and 24. In Ref. 23, it is argued that, in general, there are unitary operators $U(t)$ that diagonalize the instantaneous Hamiltonian:

$$H(t)=U(t)H_D(t)U(t)^\dagger. \tag{31}$$

In view of Eq. (3), one has

$$U(t)=U(x(t)), \tag{32}$$

where

$$x(t)=(x^i(t))\in\mathcal{G}-\{0\}=\mathbb{R}^d-\{0\}, \tag{33}$$

are the points of the loop in the parameter space. In fact, one can show that the parameter space "is not" $\mathbb{R}^d-\{0\}$, but a submanifold of this space, namely the flag manifold G/T.

To see this, let me first introduce the root system of $\mathcal{G}$ associated with Υ and the corresponding Cartan decomposition:

$$\mathcal{G}_{\mathbb{C}}=\Upsilon_{\mathbb{C}}\oplus_\alpha\mathcal{G}_\alpha, \tag{34}$$

where the subscript $\mathbb{C}$ means *complexification* and α stand for the roots. Let l denote the rank of $\mathcal{G}$, $\{H_i\}_{i=1,2,\ldots,l}$ and E_α be bases of Υ and $\mathcal{G}_\alpha$, respectively.[25,22,18,17] Then, one has

$$\begin{gathered}[H_i, H_j]=0, \quad [H_i, E_\alpha]\propto E_\alpha, \quad [E_\alpha, E_{-\alpha}]\propto H_\alpha\in\Upsilon,\\ [E_\alpha, E_\beta]\propto E_{\alpha+\beta}, \qquad \text{for} \quad \beta\neq-\alpha.\end{gathered} \tag{35}$$

Any group element can be obtained as a product of the exponentials of the generators of the algebra. In particular,

$$U(t)=\exp\left[i\sum_\alpha \chi_\alpha(t)E_\alpha\right]\exp\left[i\sum_i \chi_i(t)H_i\right]. \tag{36}$$

Since any diagonal element commutes with H_i's, it belongs to Υ. Hence, one has

$$H_D(t)=\sum_i b_i(t)H_i. \tag{37}$$

Substituting Eq. (37) in Eq. (36) and using the resulting equation to simplify Eq. (31), one obtains

$$H(t)=\exp\left(i\sum_{\alpha}\ \chi_{\alpha}(t)E_{\alpha}\right)H_D(t)\exp\left(-i\sum_{\alpha}\ \chi_{\alpha}(t)E_{\alpha}\right) \tag{38}$$

$$=\exp\left(i\sum_{\alpha>0}\ [z_{\alpha}(t)E_{\alpha}+z_{\alpha}^{*}(t)E_{-\alpha}]\right)H_D(t)$$

$$\times\exp\left(-i\sum_{\alpha>0}\ [z_{\alpha}^{*}(t)E_{\alpha}+z_{\alpha}(t)E_{-\alpha}]\right). \tag{39}$$

In Eqs. (38) and (39), $\chi_{\alpha}\in\mathbb{R}$ and $z_{\alpha}\in\mathbb{C}$ are time-dependent parameters. It is shown in Ref. 23 that, in general, the geometric phase is given in terms of χ_{α}'s, or alternatively in terms of z_{α}'s, and it does not depend on $H_D(t)$. It is not difficult to see that indeed χ_{α} correspond to the coordinates of the points of the flag manifold G/T. Alternatively, one can use the complex coordinates z_{α}. This is reminiscent of the fact that G/T has a canonical complex structure.[17] This completes the proof of the claim that the true parameter space of the system described by (30) is G/T, or a submanifold of G/T. I will come back to this point in Sec. VI. The fact that G/T can be viewed as embedded in $\mathscr{G}$ is useful because it allows one to work with the global Cartesian coordinates systems on $\mathscr{G}=\mathbb{R}^d$.[24] A natural embedding of G/T is provided by taking a regular (nondegenerate) element H_0 of Υ and considering the adjoint action of G on $\mathscr{G}$. The orbit corresponding to H_0 is a copy of G/T. Thus, one might note that in Eq. (30),

$$x=(x^i)\in G/T\subset\mathbb{R}^d. \tag{40}$$

The fact that the phase information is encoded in $U(t)$ of Eq. (31) can be used to simplify the problem, namely one can restrict to the case where the $H_D(t)=H_D(0)=H_0$ is kept constant, i.e.,

$$H_D=\sum_i\ b_iH_i=:H_0\in\Upsilon,\quad b_i=\text{const.} \tag{41}$$

The Hilbert space $\mathscr{H}$ of the quantum state vectors provides the representation space. It can be decomposed into irrep. spaces. I shall assume that $\mathscr{H}$ (or the subspace of $\mathscr{H}$ relevant to the geometric phase) corresponds to an irrep. with maximal weight Λ.[18] The weights are the simultaneous eigenvectors of H_i's.[25] They are conveniently denoted by $|\lambda_1,...,\lambda_l\rangle$, or collectively by $|\lambda\rangle$, where

$$H_i|\lambda\rangle=\lambda_i|\lambda\rangle,\quad \forall i=1,\ldots,l. \tag{42}$$

Clearly, the weight vectors $|\lambda\rangle$ are the eigenstate vectors of the initial Hamiltonian. Here, I have set $U(0)=1$ in Eq. (31).[23] In general, this can be achieved by appropriately choosing the maximal torus T. Thus, one has

$$H(x(0))=H_D=H_0 \tag{43}$$

and

$$H_D|\lambda\rangle=\sum_{i=1}^{l}\ b_i\lambda_i|\lambda\rangle. \tag{44}$$

Making the dependence of $H_D(H_0)$ on the initial point $x_0:=x(0)$ explicit, one can write Eq. (44) in the form

$$H_0(x_0)|\lambda,x_0\rangle = E_\lambda(x_0)|\lambda,x_0\rangle, \quad E_\lambda(x_0) := \sum_{i=1}^{l} b_i \lambda_i(x_0). \tag{45}$$

The weight vectors $|\lambda,x_0\rangle$ are precisely the eigenvectors $|n,x_0\rangle$ of the instantaneous Hamiltonian $H_0(x_0)$. Since x_0 can be chosen arbitrarily, one can simply drop the subscript "0," i.e., replace x_0 by x and $H_0(x_0)$ by $H(x)$.

The BS line bundle, in this case, is obtained as the pullback bundle from the universal classifying bundle E,

$$L_\lambda^{\mathrm{BS}} := f^*(E), \tag{46}$$

induced by the map

$$f: M \in x \to |\lambda,x\rangle\langle\lambda,x| \in \mathcal{P}(\mathcal{H}) \subset \mathbb{C}P^\infty.$$

Recalling some basic facts about the flag manifolds and their relation to projective spaces,[18] one finds that, in fact, L_λ^{BS} corresponds to the line bundle L_Λ of the BWB theorem, if the weight vector $|\lambda,x_0\rangle$ is chosen to be the maximal weight Λ of the representation. First, let us recall[18,17] that flag manifolds are projective varieties, i.e., there exist embeddings of M into $\mathbb{C}P^\infty$,

$$i: M \hookrightarrow \mathbb{C}P^\infty. \tag{47}$$

Indeed, one can obtain $M=G/T$ as a unique closed orbit of the action of G on $\mathcal{P}(\mathbb{C}^{N+1}) = \mathbb{C}P^N$, for some $(N+1)$-dimensional irrep. (Ref. 18, Sec. 23.3). The line bundle L_Λ is then the restriction (pullback under the identity map) of E:

$$L_\Lambda = i^*(E). \tag{48}$$

Let $|v_0\rangle$ be a nonzero vector in the representation (Hilbert) space of the Λ representation of $G, G_{\mathbb{C}}$ be the complexification of G, and consider the map

$$\Phi: G_{\mathbb{C}} \to \mathcal{P}(\mathcal{H}),$$

defined by

$$\Phi(\tilde{g}) := [U(\tilde{g})|v_0\rangle] = U(\tilde{g})|v_0\rangle\langle v_0|U(\tilde{g})^\dagger. \tag{49}$$

Here $U(\tilde{g})$ is the representation of $\tilde{g} \in G_{\mathbb{C}}$ and $[U(\tilde{g})|v_0\rangle]$ denotes the ray passing through $U(\tilde{g})|v_0\rangle$. Φ is clearly not one to one. Let P be the closed subgroup of $G_{\mathbb{C}}$ defined by

$$P := \{\tilde{h} \in G_{\mathbb{C}} \ : \ U(\tilde{h})|v_0\rangle = c|v_0\rangle, \quad \text{for some} \ \ c \in \mathbb{C} - \{0\}\}. \tag{50}$$

By construction the map Φ induces a one-to-one map on $G_{\mathbb{C}}/P$:

$$\hat{\Phi}: G_{\mathbb{C}}/P \to \mathcal{P}(\mathcal{H}). \tag{51}$$

Now, let us choose

$$|v_0\rangle := |\Lambda,x_0\rangle, \tag{52}$$

and denote by B the *Borel subgroup* of $G_{\mathbb{C}}$ generated by H_i and $E_{\alpha>0}$. Then, $B \subset P$ and consequently $G_{\mathbb{C}}/P$ is a compact submanifold (subvariety) of $G_{\mathbb{C}}/B$. However, one has the identity

$$G_{\mathbb{C}}/B = G/T,$$

where by equality I mean the diffeomorphism of homogeneous spaces.[17] Thus, in general, $G_{\mathbb{C}}/P\subset G/T$.

The extreme case is when $P=B$, i.e., $M=G_{\mathbb{C}}/P=G/T$. However, in general, B may be a proper subgroup of P, in which case the parameter manifold can be restricted to the submanifold $G_{\mathbb{C}}/P$ of G/T. This depends on the representation, i.e., on Λ.

Let us consider the general case, i.e., $M=G_{\mathbb{C}}/P$. The basic vectors $|\lambda,x\rangle$ are parametrized by the points of $G_{\mathbb{C}}/P\subset G/T$ and the map f of (16) becomes

$$f:G_{\mathbb{C}}/P\ni x\rightarrow|\lambda,x\rangle\langle\lambda,x|\in\mathscr{P}(\mathscr{H}). \tag{53}$$

In view of the fact that $G_{\mathbb{C}}/P\subset G/T$, one may work with the representative of $x=[g]\in G/T$ rather than $x=[\tilde{g}]\in G_{\mathbb{C}}/P$ for the parameters x. The next logical step is to compare the map $\hat{\Phi}$ of (51) with f. Let $x\in M\subset G/T$; then every eigenstate vector $|\lambda,x\rangle$ can be obtained by the action of G on a nonzero vector. In particular, there is a $g_x\in G$ such that

$$|\lambda,x\rangle=U(g_x)|\lambda,x_0\rangle. \tag{54}$$

Combining Eqs. (52), (53), (54), and specializing to $\lambda=\Lambda$, one finds

$$f(x)=U(g_x)|v_0\rangle\langle v_0|U(g_x)=[U(g_x)|v_0\rangle]. \tag{55}$$

Recalling the procedure according to which x is assigned to represent the parameter (40) of the system (30), one can identify $[g_x]\in G_{\mathbb{C}}/P\subset G/T$ with x, i.e.,

$$U(g_x)\equiv U(x),$$

and consequently,

$$f(x)=[U(x)|v_0\rangle]=\hat{\Phi}(x). \tag{56}$$

For the special case of $P=B$, the map $\hat{\Phi}$ becomes the map i of (47). Thus, according to Eqs. (48) and (56), the following identity is established:

$$L_\Lambda=f^*(E). \tag{57}$$

Equation (57) is valid generally, i.e., even when $P\neq B$. In this case, $M=G_{\mathbb{C}}/P$ is a proper submanifold of G/T, and the role of the embedding i of Eq. (47) is played by

$$i':M\hookrightarrow G/T\overset{i}{\hookrightarrow}\mathbb{C}P^\infty.$$

Comparing Eq. (57) with Eq. (46), one arrives at the desired result, namely that the bundle L_Λ of the BWB theorem is identical to the BS bundle $L_\Lambda^{\rm BS}$. In particular, the dimension of the irrep., i.e., the Hilbert space $\mathscr{H}$ is given by the number of the linearly independent holomorphic sections of $L_\Lambda^{\rm BS}$. The latter is a topological invariant of $L_\Lambda^{\rm BS}$.

It is well known that the topology of a complex line bundle is uniquely determined by its first Chern class $\hat{c}_1$.[26,5] $\hat{c}_1$ is represented by a closed differential two-form on M. It can be characterized by a set of $[p:=\dim H_2(M,\mathbb{Z})]$ integers by integrating it over p compact two-dimensional submanifolds of M, which are called the 2-cells of M. For example, if $G=\mathrm{SU}(2)$, $M=S^2$ and the space S^2 is the only 2-cell. Therefore, $\hat{c}_1$ is determined by a single integer c_1 via Eq. (23).

In general, the following modification of Eq. (23) provides the necessary integers,

$$c_1^a=\hat{c}_1(\sigma_a):=\frac{i}{2\pi}\int_{\sigma_a}\Omega, \tag{58}$$

where σ_a is the ath 2-cell $(a=1,\ldots,p)$, c_1^a is the first Chern number associated with σ_a, and Ω is the curvature two-form of the line bundle.

For the case of the BWB–BS line bundle, c_1^a determine the irreps. On the other hand, the irreps. are given by the maximal weight Λ of the representation. The latter can be written as a linear combination of the so-called *fundamental weights* (Ref. 18, Sec. 14.1), with non-negative integer coefficients. Let us denote these by Λ_b, $b=1,\ldots,l$. Then,

$$\Lambda=\sum_{b=1}^{l} k_b\Lambda_b, \quad k_b\in\mathbb{Z}^+\cup\{0\}. \tag{59}$$

This means that to determine the k_b's and hence the irrep. one needs precisely l "independent" first Chern numbers. These are obtained by integrating (58) over the 2-cells of G/T. The 2-cells are l copies of S^2 that correspond to the canonical SU(2) subgroups of G. These are generated by the triplets of the generators $(E_\alpha,E_{-\alpha},H_\alpha)$, where α's are the l simple roots of $\mathscr{G}$, and E_α and H_α are as in Eq. (35). Denoting these $SU(2)$ subgroups and their maximal tori by G_a and T_a, respectively, the 2-cells are given by

$$\sigma_a:=G_a/T_a=SU(2)/U(1)=S^2. \tag{60}$$

The restriction of the curvature two-form Ω on σ_a yields Berry's curvature two-form.[3] Integrating these two-forms on σ_a gives rise to l identities of the form (24). Incidentally, in view of the relevance of the system of Eq. (18) to magnetic monopoles[21] (30) corresponds to a generalized magnetic monopole whose charge has a vectorial character with integer components. I shall return to the discussion of monopoles in Sec. VI.

IV. BERRY'S CONNECTION AND THE RIEMANNIAN GEOMETRY OF THE PARAMETER MANIFOLD

One of the rather interesting facts about the geometric phase is that the AA connection $\mathscr{A}$ is related to the Fubini–Study metric on the projective space $\mathbb{C}P^N$.[27] In the language of fiber bundles, the Riemannian geometry of a manifold X means the geometry of its tangent bundle TX. In particular, the Riemannian metric (the Levi–Civita connection) is a metric (resp., a connection) on TX. The statement that the AA connection is related to the Riemannian geometry of $\mathbb{C}P^N$ is equivalent to say that the universal (AA) bundle,

$$E:\mathbb{C}\rightarrow E\rightarrow\mathbb{C}P^N,$$

is related to the tangent bundle,

$$T\mathbb{C}P^N:\mathbb{C}^N\rightarrow T\mathbb{C}P^N\rightarrow\mathbb{C}P^N.$$

This is easy to show topologically. The precise relation is demonstrated in the form of the following identity:

$$\mathrm{Det}[T\mathbb{C}P^N]=E^*\otimes E^*, \tag{61}$$

where Det means the determinant bundle:

$$\mathrm{Det}[T\mathbb{C}P^N]:=\underbrace{T\mathbb{C}P^N\wedge\cdots\wedge T\mathbb{C}P^N}_{N\ \text{times}},$$

$\wedge$ stands for the wedge product of the vector bundles, E^* is the dual line bundle to E, and $\otimes$ is the tensor product.[8] To see the validity of Eq. (61), it is sufficient to examine the first Chern classes of both sides. In fact, since $\mathbb{C}P^N$ has a single 2-cell, namely $\mathbb{C}P^1 = S^2$, one can simply compare the first Chern numbers. It is well known[10] that

$$c_1(E) = -1. \tag{62}$$

Furthermore, for any vector bundle V,

$$\hat{c}_1[\text{Det } V] = \hat{c}_1[V]. \tag{63}$$

Also, it is not difficult to show that

$$c_1(T\mathbb{C}P^N) = c_1(T\mathbb{C}P^1) = \chi(S^2) = 2, \tag{64}$$

where χ stands for the Euler–Poincaré characteristic. Equations (63) and (64) imply that

$$c_1[\text{Det } T\mathbb{C}P^N] = 2.$$

The last equality, together with the fact that

$$c_1(E^*) = -c_1(E)$$

and Eq. (62), are sufficient to establish the validity of Eq. (61).

The existence of this relationship between the AA connection and the Riemannian metric on $\mathbb{C}P^N$ has triggered the investigation of a similar pattern in the BS approach.[28] In Ref. 28, the authors discuss the case of a general Hamiltonian with a dynamical group G and a parameter space G/H, where H is a closed subgroup of symmetries of the Hamiltonian. The analysis presented above seems to include all these cases. In the following section, I will show that the system of Eq. (30) has a universal character. In other words, all the cases discussed in Ref. 28 can be reduced to the one given by (30). In all these cases the parameter space, G/H, is a submanifold of $FU(m) := U(m)/T^m$, $T^m := [U(1)]^m$, which is itself embedded into $\mathbb{C}P^\infty$. Hence, the results of Ref. 28 are expected because (i) the BS bundle (connection) is the pullback (restriction) of the universal bundle E; and (ii) E is related to $T\mathbb{C}P^N$, via Eq. (61).

V. REDUCTION OF THE NONADIABATIC PHASE TO THE ADIABATIC PHASE FOR THE CRANKED HAMILTONIANS

Let us consider an arbitrary $m\times m$ Hamiltonian H acting on $\mathscr{H} = \mathbb{C}^m$. H can be viewed as an element of the (real) vector space of all complex $m\times m$-dimensional Hermitian matrices. It is very easy to compute the real dimension of this space and find out that it is equal to m^2. Thus, H can be written as a linear combination of m^2 linearly independent Hermitian matrices. Incidentally, the generators J_i of $U(m)$ form a set of m^2 such matrices. This simply indicates that one can always express H in the form of Eq. (30). This may be seen as a realization of the Peter–Weyl theorem.[19] The particular representation of H given by Eq. (30) with $G = U(m)$ for some $m \in \mathbb{Z}^+$ might not be a practical choice. For example, the quadratic Hamiltonian,

$$H = \sum_{i,j=1}^{3} Q_{ij}\sigma_i \otimes \sigma_j,$$

with σ_i being Pauli matrices,[28,29] is more manageable in this form than in the form of Eq. (30), with J_i chosen to be the generators of $U(4)$. However, in principle, one can always use the linear representation, Eq. (30).

Actually, one can use the generators of $SU(m)$ rather than $U(m)$. This is emphasized in Ref. 23. It can be directly justified by recalling that the (m^2-1) generators of $SU(m)$ are also linearly independent, and these together with the $(m\times m)$ identity matrix I provide a basis for the space of $(m\times m)$ Hermitian matrices. The Hamiltonian H can then be written as a linear combination in this basis. Clearly, the term proportional to I does not contribute to the geometric phase. This is often used as an indication of the geometric nature of Berry's phase.[30]

An advantage of the linear representation is that it allows one to use the knowledge about the universal bundles and BWB theorem directly. In particular, in some cases, it is possible to obtain the nonadiabatic analog of the BS line bundle and the connection A. The first example of this is presented in Ref. 5. In this section, I will show that since the above argument does not refer to the adiabaticity of the system, one can always reduce the Hamiltonian to the linear form. Moreover, if the time dependence of the corresponding linear Hamiltonian is realized by cranking of the initial Hamiltonian along a fixed direction,[24] then one can obtain a nonadiabatic analog $\tilde{A}$ of Berry's connection A as a pullback connection one-form. The geometric phase is then identified with the associated holonomy of the loops in the space of parameters. This is remarkable because it means that, as far as the geometric phase is concerned, one does not need the full solution of the Schrödinger equation. The essential ingredient is the function F that induces $\tilde{A}$ as a pullback one-form from the adiabatic connection one-form A.

Wang[24] has presented a procedure that essentially computes F. Nevertheless, he does not even label this function, nor does he implement the idea of universal bundles. Let us see how the conditions introduced in Ref. 5 are realized in for cranked Hamiltonians. These conditions are the following.

(1) The cyclic states are the eigenstates of a Hermitian operator $\tilde{H}$ that depends parametrically on the points of the parameter manifold M, i.e., the cyclic states are eigenstates of $\tilde{H}(x_0)$ with $x_0=x(t=0)$.

(2) $\tilde{H}$ is related to the Hamiltonian according to

$$\tilde{H}(x)=H(F(x))=(HoF)(x), \tag{65}$$

where $F:M\to M$ is some smooth function, such that in the adiabatic limit, F approaches the identity map.

Let us first see how the first condition is fulfilled for any periodic Hamiltonian. According to a result of Floquet theory,[31] the time evolution operator for any periodic Hamiltonian is of the form

$$\mathscr{U}(t)=Z(t)e^{it\tilde{H}}, \tag{66}$$

where $\tilde{H}$ is a time-independent Hermitian operator and Z is a periodic unitary operator with the same period as the Hamiltonian, i.e.,

$$Z(t+T)=Z(t), \quad Z(0)=1. \tag{67}$$

Clearly, one has

$$\mathscr{U}(T)=e^{iT\tilde{H}}, \tag{68}$$

which justifies the first condition. The second condition can be seen to hold for the cranked Hamiltonians, either by referring to the work of Wang[24] or following the argument used in the discussion of the transformation of the Hamiltonian into the linear form. The latter is quite straightforward. One simply starts by realizing that since $\tilde{H}$ is Hermitian, it can also be written in the linear form:

$$\tilde{H}(x_0)=\sum_{i=1}^{d}\tilde{x}_0^i J_i\,, \tag{69}$$

where $\tilde{x}_0:=(\tilde{x}_0^i)\in M$ must depend on the Hamiltonian (30), and consequently on $C\subset M$. However, for the cranked Hamiltonians the time dependence of the Hamiltonian is governed by the action of a one-parameter subgroup of G, i.e., the operator $U(t)$ of Eq. (32) is given by

$$U(t):=\exp[i\omega t n_\alpha E_\alpha],\quad \text{with}\;\; n_\alpha=\text{const},$$

where ω and (n_α) are called the cranking rate and direction, respectively. It is clear that for such systems $\tilde{x}_0$ can only depend on the initial Hamiltonian and thus on x_0. The function F is defined by

$$\tilde{x}_0=:F(x_0). \tag{70}$$

The only problem is that in some cases, depending on the value of the slowness parameter $\nu(\omega)$, F may be discontinuous or even multivalued. This happens in the case of Eq. (18) for $\nu=\omega/b=1$. But in the generic case F is smooth and the second condition holds as well. The nonadiabatic analog of the BS line bundle is then given by

$$\tilde{L}:=F^*(L). \tag{71}$$

It is endowed with the nonadiabatic connection one-form,

$$\tilde{A}:=F^*(A). \tag{72}$$

For completeness, let me briefly review the arguments of Ref. 5, which lead to Eqs. (71) and (72). The basic idea is that the existence of $\tilde{H}$ that satisfies Eq. (69) allows one to imitate Berry's treatment of the adiabatic systems. The energy eigenstate vectors $|n,x\rangle$ are replaced by the eigenstate vectors $|\tilde{n},x\rangle$ of $\tilde{H}(x)$. In view of Eq. (65), these are given by

$$|\tilde{n},x\rangle=|n,\tilde{x}\rangle=|n,F(x)\rangle. \tag{73}$$

The nonadiabatic line bundle $\tilde{L}$ is obtained from the universal line bundle E via the nonadiabatic analog of the map f of Eq. (14). Denoting the latter by $\tilde{f}:M\rightarrow\mathscr{P}(\mathscr{H})$, one has

$$\tilde{f}(x):=|\tilde{n},x\rangle\langle\tilde{n},x|=|n,F(x)\rangle\langle n,F(x)|=(foF)(x).$$

Then, using the functorial property of the pullback operation, one shows that

$$\tilde{L}=\tilde{f}^*(E)=(foF)^*(E)=(F^*of^*)(E)=F^*(L), \tag{74}$$

where in the last equality Eq. (15) is used. This proves Eq. (71). The proof of Eq. (72) is identical. An important observation is that unlike $|n,x_0\rangle$, the initial state vectors $|\tilde{n},x_0\rangle$ undergo exact cyclic evolutions.

VI. MORE ON PARAMETER SPACES AND MONOPOLES

In the discussion of the the relation between the BS connection and the Riemannian structure on the parameter space, the parameter space is taken to be $M=G/H$, for some arbitrary closed subgroup H of G.[28] It can be shown that all these cases are included in the analysis of the linear system Eq. (30).

In Sec. III, I argued that depending on the (maximal weight Λ of the) irrep. of G,M is of the form $G_{\mathbb{C}}/P\subset G/T$, where P is defined by Eq. (50). Let us consider the Weyl chamber $\mathscr{W}$ of Υ^*

with respect to which the positive and the negative roots are distinguished.[18] If Λ happens to lie on at least one of the walls of $\mathscr{W}$, then B is a proper subgroup of P, otherwise $P=B$. The universal character of the linear Hamiltonian is also realized, in that all the homogeneous spaces of G can be obtained as $G_{\mathbb{C}}/P$ by choosing Λ appropriately. In fact, this is the basic idea of the classification of the compact homogeneous spaces of semisimple Lie groups. Therefore, in principle, one should be able to reproduce the results of[28] using the relation of Berry's phase to the theory of universal bundles.

Let us consider the group $G=\mathrm{SU}(3)$ in its defining (standard) representation. SU(3) is of rank $l=2$. So any irrep. is given by two integers. The standard representation is itself a fundamental representation, namely $(k_1=1,\ k_2=0)$.[18] The maximal weight is on a wall of $\mathscr{W}$ and the Borel subgroup of upper triangular matrices in $SL(3,\mathbb{C})=\mathrm{SU}(3)_{\mathbb{C}}$ is a proper subgroup of P. The subgroup P of $SL(3,\mathbb{C})$ consists of the elements of the form

$$\begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix},$$

where $*$ are complex numbers.[18] The parameter space is $M=SL(3,\mathbb{C})/P=\mathrm{SU}(3)/\mathrm{U}(2)=\mathbb{C}P^2=\mathscr{P}(\mathscr{H})$. It is interesting to see that in this case the parameter space M and projective Hilbert space $\mathscr{P}(\mathscr{H})$ are identical. In fact, this is true for all $\mathrm{SU}(N+1)$ groups. The defining representation corresponds to $(k_1=1,\ k_2=\cdots=k_N=0)$ and the parameter space is $M=\mathrm{SU}(N+1)/\mathrm{U}(N)=\mathbb{C}P^N=\mathscr{P}(\mathscr{H})$. Therefore, the inducing map f maps $\mathbb{C}P^N$ to itself for all $N>1$.

The situation is different for the octet representation of SU(3). In this case one has $k_1=k_2=1$. Λ lies in the interior of $\mathscr{W}$, $P=B$, and the parameter space is the full flag manifold $M=\mathrm{SU}(3)/\mathrm{U}(1)\times\mathrm{U}(1)$. The map f maps M into $\mathscr{P}(\mathscr{H})=\mathbb{C}P^7$. [Note that this representation is eight dimensional, i.e., the representation space for $SL(3,\mathbb{C})$ is $\mathbb{C}^8$. Hence, $\mathscr{H}=\mathbb{C}^8$.]

For $G=\mathrm{SU}(2)$, it is well known that the system of Eq. (18) is related to the magnetic monopoles.[21] The relation of monopoles to the gauge theories and their generalization to arbitrary compact semisimple gauge groups have been studied in the late 1970s.[20] These generalized monopoles are called *non-Abelian* or *multimonopoles* for general groups and *color monopoles* for $SU(3)$.[32] They are topologically classified by an associated set of l integers, where l is the rank. These are called the *topological charges* of the monopole and they are defined as elements of the second homotopy group $\pi_2(G/H)$, where H is the group of the symmetries of a ground state of the Higgs fields (a minimum of Higgs potential).[20] For $G=SU(3)$, there are two possibilities. Either

$$\text{(I)}\quad H=U(2) \quad \text{or} \quad \text{(II)}\quad H=T=U(1)\times U(1).$$

These cases have been studied in almost every article written on this subject, e.g. see Refs. 33, 20 and references therein.

If G is simply connected, then a result of algebraic topology indicates that

$$\pi_2(G/H)=\pi_1(H).$$

Applying this result to $G=\mathrm{SU}(3)$, one finds

$$\text{(I)}\quad \pi_2(\mathrm{SU}(3)/\mathrm{U}(2))=\pi_1(\mathrm{U}(2))=\mathbb{Z},$$

$$\text{(II)}\quad \pi_2(\mathrm{SU}(3)/\mathrm{U}(1)\times\mathrm{U}(1))=\pi_1(\mathrm{U}(1)\times\mathrm{U}(1))=\mathbb{Z}\oplus\mathbb{Z}.$$

Thus, for (I) and (II) one has, respectively, one and two topological charges. This is precisely the case with the topological charges of the geometric phase defined earlier. The same correspondence holds for arbitrary compact, connected semisimple Lie groups.

The possible relevance of the topological charges of monopoles to the representations of the group have been conjectured by Goddard *et al.*.[34] Although the analysis of the present paper does not prove their conjecture, it provides a formula for the topological charges as integrals of the first Chern class, defined by Berry's connection, over the 2-cells σ_a of Sec. III. There is a simple topological explanation for the correspondence of the topological charges of the monopoles and those of the geometric phase. This can be summarized in the identity

$$\pi_2(G/H)=H_2(G/H,\mathbb{Z}),$$

where $H_2(\cdot,\mathbb{Z})$ denotes the second homology group. This identity is a consequence of *Hurewicz theorem*,[35] where one uses the fact that $\pi_1(G/H)=H_1(G/H)=0$. The 2-cells σ_a are indeed the generators of $H_2(G/T,\mathbb{Z})$. For $H\neq T$, some of them may be smashed to a point, as is the case for $G=\mathrm{SU}(3)$ and $H=\mathrm{U}(2)$.

VII. CONCLUSION

The relationship between the phenomenon of Berry's phase and the Borel–Weil–Bott theorem is a direct consequence of the application of the universal bundles in the Aharonov–Anandan definition of the geometric phase. This relationship is appealing, not only because it links quantum mechanics to yet another central mathematical result, but also because it offers a better understanding of the theoretical foundations of geometric phases. The implications of the fact that the A–A bundles are indeed the universal bundles of mathematics for the study of nonadiabatic phases is a typical indication of the importance of this observation.

The identification of the mathematical structures used in the holonomy interpretations of the geometric phase with those employed in the Borel–Weil–Bott theorem sheds light on a number of unresolved issues. Among these are the determination of the appropriate parameter space and the relation between the geometry of the parameter space and the geometric structure of the phase. The BWB theorem leads to the introduction of a set of topological charges, which determine the topology of the BS line bundles and thus encompass all the topological content of the phase. These charges seem to be related to, if not identical with, the topological charges of non-Abelian monopoles. The integral nature of these charges is a consequence of the topological properties of the first Chern class. The latter is essentially the reason for the quantization of the charges of the monopoles.

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Note added in proof: The material presented in this paper is also relevant to the subject of Ref. 36. The author wishes to thank the referee for informing him of this reference.

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