Journal of Physics A: Mathematical and Theoretical

PAPER

The physics of exceptional points

To cite this article: W D Heiss 2012 *J. Phys. A: Math. Theor.* **45** 444016

J. Phys. A: Math. Theor. **45** (2012) 444016 (11pp) doi:10.1088/1751-8113/45/44/444016

The physics of exceptional points

W D Heiss

National Institute for Theoretical Physics, Stellenbosch Institute for Advanced Study, Stellenbosch 7602, South Africa

E-mail: dieter@physics.sun.ac.za

Received 8 March 2012, in final form 23 April 2012
Published 23 October 2012
Online at stacks.iop.org/JPhysA/45/444016

Abstract

A short résumé is given about the nature of exceptional points (EPs) followed by discussions about their ubiquitous occurrence in a great variety of physical problems. EPs feature in classical as well as in quantum mechanical problems. They are associated with symmetry breaking for $\mathcal{PT}$-symmetric Hamiltonians, where a great number of experiments has been performed, in particular in optics, and to an increasing extent in atomic and molecular physics. EPs are involved in quantum phase transition and quantum chaos; they produce dramatic effects in multichannel scattering, specific time dependence and more. In nuclear physics, they are associated with instabilities and continuum problems. Being spectral singularities they also affect approximation schemes.

This article is part of a special issue of *Journal of Physics A: Mathematical and Theoretical* devoted to 'Quantum physics with non-Hermitian operators'.

PACS numbers: 03.65.Vf, 03.65.Xp, 31.15.−p, 02.30.−f, 02.30.Tb

1. Introduction

Singularities of functions describing analytically observable quantities have always been in the scrutiny of theoretical investigations. For instance, the structures of measured cross sections are usually associated with pole terms in the complex energy plane of the scattering amplitudes. In turn, these pole terms are associated with specific boundary conditions of solutions of, say, the Schrödinger equation [1]. Another example is the pattern of spectra when plotted versus an external strength parameter, say, of a magnetic field; it usually shows the phenomenon of level repulsion, often associated with quantum chaos [2]. When such spectra are continued into the complex plane of the strength parameter, one encounters a different type of singularity where two repelling levels are connected by a square root branch point. If for a real strength parameter the Hamiltonian is Hermitian, the branch points always occur at complex parameter values, thus rendering the continued Hamiltonian non-Hermitian. As a consequence, the well-known properties associated with degeneracy of Hermitian operators are no longer valid. These singularities have been dubbed *exceptional points* (EPs) by Kato [3].

A few decades ago these singularities were perceived as a mathematical phenomenon that would be ‘in the way’ of approximation schemes [4, 5] (see also the discussion in section 4 below). However, their physical significance was recognized in an early paper by Berry [6] based on an observation by Pancheratnam [7]. The specific algebraic property of the dielectric tensor (it cannot be diagonalized) brings about particular physical effects. This has been expounded in great detail in [8] for particular optical systems. As discussed below, optical systems constitute one major realm where EPs have a major bearing. Yet, originating from a parameter-dependent eigenvalue problem EPs naturally occur and can give rise to dramatic effects in a great variety of physical problems: in mechanics, electromagnetism, atomic and molecular physics, quantum phase transitions, quantum chaos and more.

In section 3, we return to these phenomena in detail. In subsection 3.1, we discuss the experimental manifestation using microwave cavities where all the mathematical properties have been confirmed to be a physical reality. The investigation and physical realization of $\mathcal{PT}$-symmetric Hamiltonians has become a major subject of theoretical and experimental endeavour. An important role is played by EPs in this research; it is dealt with in subsection 3.2. Subsection 3.3 is devoted to a short discussion of the role of EPs in atomic and molecular physics while in 3.4 recent findings in open systems and laser physics are mentioned. In subsection 3.5, we demonstrate the essential role of EPs in understanding the dramatic effects associated with a quantum phase transition and quantum chaos, while subsection 3.6 presents a short discussion of multichannel scattering and subsection 3.7 touches upon EPs of higher order. Classical mechanics and fluids are briefly dealt with in subsection 3.8. The sometimes adverse role of EPs in approximation schemes is, for historical reasons, discussed in section 4.

For the sake of completeness, we rehash the essential formal background in section 2. The last section 5 gives a summary.

2. Exceptional points

EPs occur generically in eigenvalue problems that depend on a parameter. By variation of such a parameter (usually into the complex plane), one can generically find points where eigenvalues coincide.

In the immediate vicinity of an EP, the special algebraic behaviour—as discussed below—allows a reduction of the full problem to the two-dimensional problem associated with the two coinciding levels [9]. We thus confine our discussion to the eigenvalues of a two-dimensional matrix where the direct connection of an EP and the phenomenon of level repulsion is easily demonstrated. Consider the problem

$$\begin{aligned} H(\lambda) &= H_0 + \lambda V \\ &= \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix} + \lambda \begin{pmatrix} \epsilon_1 & \delta_1 \\ \delta_2 & \epsilon_2 \end{pmatrix}, \end{aligned} \tag{1}$$

where the parameters ω_k and ϵ_k determine the non-interacting resonance energies $E_k = \omega_k + \lambda\epsilon_k,\ k = 1, 2$. We may choose all complex parameters and require $[H_0, V] \neq 0$ to ensure the problem is non-trivial. Owing to the interaction invoked by the matrix elements δ_k, the two levels do not cross but repel each other. However, the two levels *coalesce* at specific values of λ in the vicinity of the level repulsion, that is, at the two EPs

$$\lambda_1 = \frac{-\mathrm{i}(\omega_1 - \omega_2)}{\mathrm{i}(\epsilon_1 - \epsilon_2) + 2\sqrt{\delta_1\delta_2}} \tag{2}$$

$$\lambda_2 = \frac{-\mathrm{i}(\omega_1 - \omega_2)}{\mathrm{i}(\epsilon_1 - \epsilon_2) - 2\sqrt{\delta_1\delta_2}}. \tag{3}$$

For $\delta_k \neq 0$, the energy levels have a square root singularity as a function of λ and read

$$E_{1,2}(\lambda) = \tfrac{1}{2}\big(\omega_1 + \omega_2 + \lambda(\epsilon_1 + \epsilon_2) \pm \sqrt{(\epsilon_1 - \epsilon_2)^2 + 4\delta_1\delta_2}\sqrt{(\lambda - \lambda_1)(\lambda - \lambda_2)}\big), \tag{4}$$

and the eigenvalues at the EPs are

$$E(\lambda_{1,2}) = \frac{\epsilon_1\omega_2 - \epsilon_2\omega_1 \mp \mathrm{i}\sqrt{\delta_1\delta_2}(\omega_1 + \omega_2)}{\epsilon_1 - \epsilon_2 \mp 2\mathrm{i}\sqrt{\delta_1\delta_2}}. \tag{5}$$

We use the term *coalesce* as the pattern is distinctly different from a degeneracy usually (but not only) encountered for Hermitian operators. Note that $H(\lambda)$ is Hermitian only when ω_k, ϵ_k and λ are real and $\delta_1 = \delta_2^*$. At the EP, the difference between a degeneracy and a coalescence is clearly manifested by the occurrence of only *one* eigenvector instead of the familiar two in the case of a genuine degeneracy. Using the bi-orthogonal system for a non-Hermitian matrix, the (only one) right-hand eigenvector reads at $\lambda = \lambda_1$ (up to a factor)

$$|\phi_1\rangle = \begin{pmatrix} \dfrac{+\mathrm{i}\delta_1}{\sqrt{\delta_1\delta_2}} \\ 1 \end{pmatrix} \tag{6}$$

and at $\lambda = \lambda_2$:

$$|\phi_2\rangle = \begin{pmatrix} \dfrac{-\mathrm{i}\delta_1}{\sqrt{\delta_1\delta_2}} \\ 1 \end{pmatrix}, \tag{7}$$

while the corresponding left-hand eigenvector is at λ_1 and λ_2:

$$\langle\tilde{\phi}_1| = \left(\frac{+\mathrm{i}\delta_2}{\sqrt{\delta_1\delta_2}}, 1\right) \tag{8}$$

$$\langle\tilde{\phi}_2| = \left(\frac{-\mathrm{i}\delta_2}{\sqrt{\delta_1\delta_2}}, 1\right), \tag{9}$$

respectively. Note that the norm—that is, the scalar product $\langle\tilde{\phi}_k|\phi_k\rangle,\ k = 1, 2$—vanishes, which is often referred to as self-orthogonality [10]. It is this vanishing of the norm that enables the reduction of a high-dimensional problem to two dimensions in the close vicinity of an EP [11]. The existence of only one eigenvector with vanishing norm is related to the fact that for $\lambda = \lambda_1$ or $\lambda = \lambda_2$, the matrix $H(\lambda)$ cannot be diagonalized [3]. At these points, the Jordan decomposition reads

$$H(\lambda_1) = S\begin{pmatrix} E(\lambda_1) & 1 \\ 0 & E(\lambda_1) \end{pmatrix} S^{-1} \tag{10}$$

with

$$S = \begin{pmatrix} \dfrac{\mathrm{i}\delta_1}{\sqrt{\delta_1\delta_2}} & \dfrac{2\mathrm{i}\sqrt{\delta_1\delta_2} - \epsilon_1 + \epsilon_2}{(\omega_1 - \omega_2)\delta_2} \\ 1 & 0 \end{pmatrix}. \tag{11}$$

Similar expressions hold at $\lambda = \lambda_2$. We mention that the second column of S is often referred to as an associate vector obeying $(H(\lambda_1) - E(\lambda_1))|\psi_{\mathrm{assoc}}\rangle = |\phi_1\rangle$.

If one or both δ_k vanish there is level crossing ($\lambda_1 = \lambda_2$). If only δ_1 or δ_2 vanishes, then V and thus $H(\lambda)$ cannot be diagonalized at the crossing point; therefore, the Jacobian form is non-diagonal and only one eigenvector exists at the crossing point; yet it is not an EP as there is no square root singularity in λ. We do not further pursue such a case as it appears to be of no physical interest.

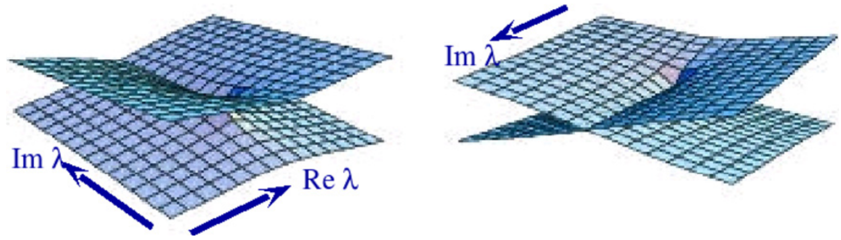

Figure 1. Perspective view of the Riemann sheet structure of two coalescing energy levels in the complex λ-plane. The assignment for the axes assumes all other parameters as real; if some or all of them are complex the sheet structure remains but is shifted and/or turned.

The square root singularity also affects Green's function and thus the scattering matrix in a particular way. At the EPs, Green's function (and scattering matrix) has a pole of second order [12] in addition to the familiar pole of first order. For the case considered here the explicit form of Green's function reads at λ_1 as

$$(E - H(\lambda_1))^{-1} = \frac{1}{E - E(\lambda_1)}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\mathrm{i}\sqrt{\delta_1\delta_2}\lambda_1}{(E - E(\lambda_1))^2}\begin{pmatrix} 1 & \mathrm{i}\sqrt{\frac{\delta_1}{\delta_2}} \\ \mathrm{i}\sqrt{\frac{\delta_2}{\delta_1}} & -1 \end{pmatrix} \tag{12}$$

and similarly at λ_2. The first term resembles the conventional expression at a non-singular point of the spectrum; the second term is a consequence of the singular spectral point. This term can give rise to dramatic effects near the EP [13], and at the EP special effects are generated from the interference of the two terms when the square of the modulus of the scattering matrix is considered.

The square root behaviour as given by equation (4) has further physical consequences as discussed in the following sections. Here, we list the major mathematical reasons giving rise to special physical phenomena.

- In the vicinity of an EP, the spectrum is strongly dependent on the interaction parameter; in fact, the derivative with respect to λ of the eigenvalues and eigenvectors is infinity at the EP.
- Encircling λ_1 or λ_2 in the λ-plane will interchange the two energy levels (see figure 1).
- At a finite distance from the EP there are two linearly independent eigenvectors $|\psi_1\rangle$ and $|\psi_2\rangle$ (and the left-hand companions $\langle\tilde{\psi}_1|$ and $\langle\tilde{\psi}_2|$) with the normalization $\langle\tilde{\psi}_k|\psi_k\rangle = 1$. Enforcing the normalization into the EP (where $\langle\tilde{\psi}_k|\psi_k\rangle$ vanishes), the components of the eigenvectors tend to infinity as $\sim 1/(\lambda - \lambda_k)^{1/4}$. Repeated encircling—say counterclockwise—of an EP generates the pattern for the normalized eigenvectors
$$|\psi_1\rangle \to -|\psi_2\rangle \to -|\psi_1\rangle \to |\psi_2\rangle \to |\psi_1\rangle, \tag{13}$$
from which the pattern for $|\psi_2\rangle$ follows accordingly [9, 14]; the fourth root behaviour is clearly seen as four rounds are needed to reach the initial sheet. Note that these relations imply a kind of chiral behaviour: going clockwise instead of counterclockwise gives a different sign. We note that these relations are presented inconsistently in [15].
- When the eigenfunctions $|\psi_1\rangle$ and $|\psi_2\rangle$ coalesce as given in (6) and (7) (here, we normalize the vectors by setting one component equal to unity), they become, for $\delta_1 = \delta_2$, independent of parameters and assume the form
$$\begin{pmatrix} \pm\mathrm{i} \\ 1 \end{pmatrix}.$$

The phase difference of $\pi/2$ between the two components is changed if, for instance, time reversal symmetry is broken by choosing complex $\delta_1 = \delta_2^*$ [16, 17]. Note further that λ_k must be complex for Hermitian operators H_0 and V, in which case the two EPs occur at complex conjugate values, i.e. $\lambda_1 = \lambda_2^*$. In general λ_1 or λ_2 can be real [18]. Recall that an EP can be approached in the laboratory only if the corresponding energy has a non-positive imaginary part.

3. Physical effects

Many cases of particular effects have been reported in the literature during the past ten years. We here discuss only some major trends and developments. While we focus the discussion upon quantum mechanical problems and optics, the ubiquitous character of EPs in any parameter-dependent eigenvalue problem makes them appear also in problems of classical mechanics and others.

3.1. Microwave cavity

Probably for the first time ever the direct encircling of the square root branch point—that is, the manifestation of the two Riemann sheets (see figure 1)—was accomplished with a microwave resonator [19].

The realization of the complex parameter λ was implemented in the laboratory by two real parameters: (i) the coupling between the two halves of the cavity and (ii) the variation of the level in one half of the cavity. In one experiment the direct approach to the EP was avoided, while the encircling was done at a close distance. All properties as listed in the previous section have been confirmed: encircling an EP swaps the energies and fourfold encirclements yield relations (13).

The chiral property of the wavefunction at the EP has been confirmed in a second experiment [20], where the phase difference of $\pi/2$ between the two components has been measured in a direct approach to an EP.

The same results have been established with two coupled electronic circuits [21].

More recent experiments implementing careful time reversal symmetry breaking by using a magnetized device at the coupling of two cavities [16] are in perfect accordance with the predictions made in [17].

Other types of experiments and/or theoretical investigations with microwave cavities are found in [22, 23] where the effects of EPs feature prominently. The study of continuous variation of the coupling between a cavity and a rubidium atom enabled the direct observation of an EP in an open quantum system [24].

3.2. $\mathcal{PT}$-symmetric Hamiltonians

It has been suggested to extend the class of traditional Hermitian Hamiltonians by a specific choice of non-Hermitian operators [25]. Hamiltonians that are symmetric under the combined operation of parity and time reversal transformation ($\mathcal{PT}$-symmetric operators) can have a real spectrum even though the operators can be non-Hermitian. If the eigenstates are also symmetric under $\mathcal{PT}$, that is, if $\mathcal{PT}|\psi_E\rangle = \text{const}|\psi_E\rangle$, the eigenvalues are real; when the symmetry is broken, that is, if the above relation does not hold, the eigenvalues are complex [26]. It turns out that the parameter values where the symmetry breaking occurs are just those values where an EP of the system appears. Since the Hamiltonians considered in this class can be non-Hermitian, the EPs can occur for real parameter values. Obviously, this particular extension of traditional quantum mechanics has attracted great interest in the literature with umpteen theoretical and experimental papers.

Before turning to a selection of these we rehash some basic features of the specific class of operators considered here. While unbounded non-Hermitian operators constitute a difficult mathematical problem in general, we here deal with a much simpler class that is well understood: the *quasi-Hermitian* operators [27]. A non-Hermitian operator $H \neq H^\dagger$ is called quasi-Hermitian if there exists a bounded Hermitian positive-definite operator Θ that ensures the relation

$$\Theta H = H^\dagger \Theta. \tag{14}$$

The relation implies that H is similarly equivalent to a Hermitian operator, in fact the operator

$$h_S = SHS^{-1} \tag{15}$$

is Hermitian when S is the positive root of Θ. The operator Θ may be viewed as a metric characterizing a different Hilbert space by defining the scalar product

$$\langle\cdot|\cdot\rangle_\Theta := \langle\cdot|\Theta\cdot\rangle, \tag{16}$$

where $\langle\cdot|\cdot\rangle$ is the usual scalar product, employing the L^2-metric being the identity. Two important observations can be made. (i) The non-Hermitian operator H is in fact Hermitian if the new metric is used for the underlying Hilbert space; note, however, that then other operators (like position or momentum) may no longer be necessarily Hermitian. There is freedom though in the choice of the metric Θ [27, 28]. (ii) In view of (15) the spectrum of H is real.

Of particular interest in our context is the fact that at a parameter value where the symmetry gets broken—at the EP—the singularity affects the metric as well; in fact, the metric ceases to exist [29]. While the occurrence of an EP has been shown to cause a great variety of physical effects, the breakdown of the metric has been associated with a speculative suggestion: that the transition time between the two states connected at the EP may be much shorter than the expected time $\sim\hbar/\Delta E$ [30].

Many theoretical papers have been published on the subject dealing with the mathematical aspects of diagonalizable non-Hermitian operators [31–34]; the list given there can only be incomplete while the quoted papers contain further pertinent references. More recent theoretical papers deal with specific subjects such as $\mathcal{PT}$-symmetry in optics [35] and nonlinear wave equations [36]. In [37] a new class of chaotic systems with dynamical localization is studied: a gain/loss parameter invokes a spontaneous phase transition from real values of the spectrum (phase of conserved symmetry) to complex values (phase of broken symmetry).

Perhaps some of the most beautiful demonstrations of $\mathcal{PT}$-symmetry breaking at an EP have been made in optical systems. Light propagation is used in distributed feedback optical structures with gain or loss regions. The EPs at the points of $\mathcal{PT}$-symmetry breaking of the Dirac Hamiltonian give rise to simple observable physical quantities such as resonance narrowing and laser oscillation [38]. In $\mathcal{PT}$-symmetric optical lattices the transition at the spectral singularity (EP) has been demonstrated [39] (see also [40–44]). More experimental work along these lines can be expected. In particular, specific quantum mechanical systems like the ones discussed in the following subsection are expected to involve aspects of $\mathcal{PT}$-symmetry.

An interesting twist is the classical analogue to quantum mechanical $\mathcal{PT}$-symmetric Hamiltonians [45]. It appears that the correspondence principle can be usefully extended into the complex domain of classical variables. The association between complex quantum mechanics and complex classical mechanics is subtle and requires a non-perturbative approach (see also [29, 46]).

Quite generally and in the spirit of $\mathcal{PT}$-symmetry, its consideration is of course not restricted to quantum systems. Here, we mention electronic circuits where the connection of instabilities in $\mathcal{PT}$-symmetric systems and EPs is investigated experimentally [47] and theoretically [48].

3.3. EPs in atomic/molecular physics, Feshbach resonances

Using Feshbach resonance techniques there are recent proposals for resonant dissociation by lasers of H_2^+ molecules or alkali dimers, where the effects of EPs are expected to feature prominently [49, 50]. Similar in spirit, a Bose–Einstein condensate of neutral atoms with induced electromagnetic attractive $(1/r)$ interaction has been discussed recently as another system allowing a tunable interaction [51]. The critical value—an EP—where the onset of the collapse of the condensate occurs is interpreted as a transition point from separate atoms to the formation of molecules or clusters [52]. In this context, we recall that level splitting by potential barriers (quantum tunneling) is associated with a coalescence of the two levels at an appropriate (complex) value of the barrier strength.

In a recent investigation, it was found that by transporting an eigenstate around an EP the final eigenstate may be different from the initial eigenstate. The precise interplay between gain/loss and non-adiabatic couplings imposes specific limitations on the observability of this flip or non-flip effect [53].

3.4. EPs in laser physics and open systems

EPs can strongly affect the above-threshold behaviour of laser systems. They are induced by pumping the laser nonuniformly [54]. In the vicinity of the EPs, where the laser modes coalesce, the effect of the singularities can explain the turning off of one laser even when the overall pump power deposited in the system is increased. The relation between EPs in closed and open gain–loss structures has been expounded recently [55]. The EPs occurring for a closed-system Hamiltonian are related to those of the scattering matrix for the corresponding open system. It has been demonstrated in detail how these seemingly different situations share a very close and elegant connection with each other.

3.5. Quantum phase transitions, chaos

The important role played by EPs in connection with quantum phase transition and also quantum chaos in many-body systems appears to be less appreciated by the 'EP community'. We here report some pertinent results. The Lipkin model [56] is a toy model often used to study quantum phase transitions of many-body systems. The interaction of the two-level model lifts or lowers a Fermion pair between the two levels. For N particles, it can be formulated in terms of the angular momentum operators and reads

$$H(\lambda) = J_z + \frac{\lambda}{N}(J_+^2 + J_-^2), \tag{17}$$

with $J_z, J_\pm$ being the N-dimensional representations of the $SU(2)$ operators. There is a phase transition at $\lambda > 1$ that moves towards $\lambda = 1$ in the thermodynamic limit ($N \to \infty$). The special interaction gives rise to an inherent symmetry: even and odd numbers k of the ordered levels E_k do not interact. The phase for $\lambda < 1$ is the 'normal' phase where the symmetry of the problem is preserved by the levels and wavefunctions. In the 'deformed' phase for $\lambda > 1$ the symmetry is broken, in that even and odd k become degenerate. Here, the role of

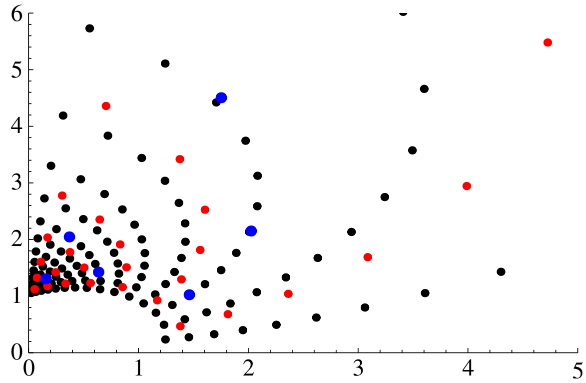

Figure 2. Exceptional points (EPs) in the complex λ-plane for the Lipkin model with $N = 8$ (blue), $N = 16$ (red) and $N = 32$ (black). The symmetry of the model yields EPs in the other quadrants: with λ_{EP} additional EPs occur at λ_{EP}^* and $-\lambda_{\text{EP}}$.

the EPs is crucial to bring about the phase transition in the spectrum [57, 58]. In figure 2, the pattern of the EPs is illustrated for low values of N. It is clearly seen how the EPs accumulate for increasing N on the real axis with the tendency to move towards the point $\lambda = 1$. The spectrum remains unaffected by singularities in the region of the normal phase while it is strongly affected around the critical point. For finite temperature, these singularities feature in the partition function as is discussed explicitly in [59]. If the model is perturbed the regular pattern of the EPs is destroyed and so is the spectrum accordingly. The onset of chaos [60] is clearly discernible in the region of the phase transition associated with a high density of EPs, while the model remains robust outside the critical region for sufficiently mild perturbation.

3.6. Special effects in multichannel scattering

Depending on a judicious choice of parameters the proximity of EPs can invoke dramatic effects in multichannel scattering such as a sudden increase of the cross section in one channel, even by orders of magnitude. In turn, the second channel is suppressed and can show a resonance curve that deviates substantially from the usual Lorentz shape [13]. Related to this behaviour is the pattern in the time domain [61]. Depending on the initial conditions, the wavefunction displays characteristic features such as very fast decay or the opposite, i.e. very long lifetime. At the EP, the time-dependent wavefunction typically has a linear term in time besides the usual exponential behaviour.

3.7. EPs of higher order

The coalescence of three levels [62] (or more) is of course possible if sufficiently many parameters are at one's disposal. For an EPN (N-levels coalescing) an N-dimensional matrix must be considered. For complex symmetric matrices $(N^2+N-2)/2$ parameters are needed to enforce the coalescence of N levels [63]. An interesting aspect is the behaviour of, say, an EP3 where two EP2 sprout from the EP3 under variation of one of the parameters; in this way, we can view an EP3 as the coalescence of two EP2 being linked by their position on one common Riemann sheet. This can be generalized allowing many combinations for larger N: the block structure of the Jordan form of the Hamiltonian gives an indication of the connectedness of the N levels. In a recent paper [64], this is investigated for EP3; a physical example for possible implementation in the laboratory is suggested. Topological properties and their group structure of higher order EPs are investigated in [65].

3.8. Classical systems

Although some of the experiments with microwave cavities as well as some of the optical systems are classical in nature, we here note in particular effects of EPs in classical [66] and fluid mechanics. Instabilities and particular behaviour of the Reynold number for a Poiseuille flow have been associated with near crossings of eigenvalues and—as is now identified—with EPs [67, 68] (see also [69]).

4. The role of EPs in approximation schemes

The well-known random phase approximation used in many-body problems yields an effective Hamiltonian that is non-Hermitian [70]. As a result, eigenvalues are not necessarily real. Depending on the strength of the, say, particle–hole interaction two real eigenvalues Ω and $-\Omega$ coalesce at $\Omega = 0$ and then move into the complex plane when the interaction is increased. Often this instability point is associated with one or more phase transitions of the underlying mean field [71]. It is an EP with all its characteristics: square root branch point in the interaction strength and the vanishing norm of the wavefunction.

A perturbative approach in shell model calculations can be hampered by singularities associated with *intruder states* [72]. These singularities are EPs where two levels coalesce thus limiting the radius of convergence of the perturbation series. In a similar vein, branch point singularities (EPs) have been identified as affecting the convergence of perturbation series in a field theoretical model [73] and for the anharmonic oscillar [74]; in these infinite dimensional cases accumulation points of EPs are encountered.

Recent approaches to model nuclei near the drip line [75] use resonance states to describe the continuum. The coalescence of two resonances can invoke specific physical effects owing to the strong increase of the associated spectroscopic factors being caused by the vanishing norm of the wavefunctions at the EP.

5. Summary

The ubiquitous occurrence of EPs in all eigenvalue problems that depend on a parameter can have significant and often dramatic effects on observables in a great variety of physical phenomena. A few decades ago, these singularities appeared as a purely mathematical feature that could cause problems in approximation schemes. It was only about fifteen years ago that their physical manifestation was demonstrated in experiments that were basically classical in nature. At present, definite theoretical and experimental proposals are found in the literature relating to atomic and molecular physics, using lasers for triggering and measuring specific transitions. In nuclear physics, where there is now great interest in open systems, that is, in nuclei on the drip line, the coalescence of resonance states is expected to produce specific effects such as enhancements of particular reactions. The tremendous upbeat of research in connection with $\mathcal{PT}$-symmetric Hamiltonians is expected to further proliferate new effects and applications of EPs also and in particular in genuine quantum mechanical problems.

References

[1] Newton R G 1967 *Scattering of Wave and Particles* (New York: McGraw-Hill)
[2] see for instance Wintgen D and Friedrich H 1986 *Phys. Rev. Lett.* **57** 571
[3] Kato T 1966 *Perturbation Theory of Linear Operators* (Berlin: Springer)
[4] Heiss W D 1970 *Nucl. Phys.* A **144** 417
[5] Moiseyev N and Friedland S 1980 *Phys. Rev.* A **22** 618

[6] Berry M V 1994 *Curr. Sci.* **67** 220
[7] Pancharatnam S 1955 *Proc. Ind. Acad. Sci.* **42** 86
[8] Berry M V and O'Dell D H J 1998 *J. Phys. A: Math. Gen.* **31** 2093
[9] Heiss W D 1999 *Eur. Phys. J.* D **7** 1
Heiss W D 2000 *Phys. Rev.* E **61** 929
[10] Moiseyev N 2011 *Non-Hermitian Quantum Mechanics* (Cambridge: Cambridge University Press)
[11] Heiss W D and Harney H L 2001 *Eur. Phys. J.* D **17** 149
[12] Hernandez E, Jaureguit A and Mondragon A 2000 *J. Phys. A: Math. Gen.* **33** 4507
[13] Heiss W D and Nazmitdinov R G 2010 *Eur. Phys. J.* D **58** 53
[14] Mehri-Dehnavi H and Mostafazadeh A 2008 *J. Math. Phys.* **49** 082105
[15] Rotter I 2009 *J. Phys. A: Math. Theor.* **42** 153001
[16] Dietz B, Harney H L, Kirillov O N, Miski-Oglu M, Richter A and Schäfer F 2011 *Phys. Rev. Lett.* **106** 150403
[17] Harney H-L and Heiss W D 2004 *Eur. Phys. J.* D **29** 429
[18] Heiss W D 2004 *J. Phys. A: Math. Gen.* **37** 2455
[19] Dembowski C, Gräf H-D, Harney H L, Heine A, Heiss W D, Rehfeld H and Richter A 2001 *Phys. Rev. Lett.* **86** 787
[20] Dembowski C, Dietz B, Gräf H-D, Harney H L, Heine A, Heiss W D and Richter A 2003 *Phys. Rev. Lett.* **90** 034101
Dembowski C, Dietz B, Gräf H-D, Harney H L, Heine A, Heiss W D and Richter A 2004 *Phys. Rev.* E **69** 056216
[21] Stehmann T, Heiss W D and Scholtz F G 2004 *J. Phys. A: Math. Gen.* **37** 7813
[22] Lee S-B *et al* 2009 *Phys. Rev. Lett.* **103** 134101
[23] Wiersig J, Kim S W and Hentschel M 2008 *Phys. Rev.* A **78** 053809
Wiersig J 2011 *Phys. Rev.* A **84** 063828
[24] Choi Y *et al* 2010 *Phys. Rev. Lett.* **104** 153601
[25] Bender C M and Boettcher S 1998 *Phys. Rev. Lett.* **80** 5243
[26] Dorey P, Dunning C and Tateo R 2001 *J. Phys. A: Math. Gen.* **34** L391
[27] Scholtz F G, Geyer H B and Hahne F J W 1992 *Ann. Phys., NY* **213** 74
[28] Geyer H B, Heiss W D and Scholtz F G 2008 *Can. J. Phys.* **86** 1195
[29] Musumbu D P, Geyer H B and Heiss W D 2007 *J. Phys. A: Math. Theor.* F **75**
[30] Bender C M, Brody D C, Jones H F and Meister B K 2007 *Phys. Rev. Lett.* **98** 040403
[31] Mostafazadeh A 2002 *J. Math. Phys.* **43** 205
Mostafazadeh A 2002 *J. Math. Phys.* **43** 2814
Mostafazadeh A 2002 *J. Math. Phys.* **43** 3944
[32] Bender C M 2007 *Rep. Prog. Phys.* **70** 947
[33] Znojil M 2006 *J. Phys. A: Math. Gen.* **39** 441
[34] Günter U, Rotter I and Samsonov B F 2007 *J. Phys. A: Math. Theor.* **40** 8815
[35] Graefe E-M and Jones H F 2011 *Phys. Rev.* A **84** 013818
[36] Bagchi B and Fring A 2008 *J. Phys. A: Math. Theor.* **41** 392004
[37] West C, Kottos T and Prosen T 2010 *Phys. Rev. Lett.* **104** 054102
[38] Longhi S 2010 *Phys. Rev. Lett.* **105** 13903
[39] Makris K G, El-Ganainy R, Christodoulides D N and Musslimani Z H 2010 *Phys. Rev.* A **81** 63807
[40] Guo A *et al* 2009 *Phys. Rev. Lett.* **103** 093902
[41] Lin Z *et al* 2011 *Phys. Rev. Lett.* **106** 213901
[42] Chong Y D, Ge L and Stone A D 2011 *Phys. Rev. Lett.* **106** 093902
[43] Mostafazadeh A 2009 *Phys. Rev. Lett.* **102** 220402
[44] Bendix O, Fleischmann R, Kottos T and Shapiro B 2009 *Phys. Rev. Lett.* **103** 030402
[45] Bender C M, Hook D, Meisinger P and Wang Q 2010 *Phys. Rev. Lett.* **104** 061601
[46] Mostafazadeh A 2006 *J. Phys. A: Math. Gen.* **39** 10171
[47] Schindler J, Li A, Zheng M C, Ellis F M and Kottos T 2011 *Phys. Rev.* A **84** 040101
[48] Kirillov O N 2012 *Phys. Lett.* A **376** 1244
Kirillov O N and Verhulst F 2010 *Z. Angew. Math. Mech.* **90** 462
[49] Lefebvre R and Atabek O 2010 *Eur. Phys. J.* D **56** 317
Atabek O *et al* 2011 *Phy. Rev. Lett.* **106** 173002
[50] Lefebvre R, Atabek O, Sindelka M and Moiseyev N 2009 *Phys. Rev. Lett.* **103** 123003
[51] Papadopoulos I, Wagner P, Wunner G and Main J 2007 *Phys. Rev.* A **76** 053604
[52] Cartarius H, Main J and Wunner G 2008 *Phys. Rev.* A **77** 013618
[53] Uzdin R, Mailybaev A and Moiseyev N 2011 *J. Phys. A: Math. Theor.* **44** 435302
[54] Liertzer M, Ge L, Cerjan A, Stone A D, Türeci H E and Rotter S 2012 *Phys. Rev. Lett.* **108** 173901

[55] Rotter S 2012 Exceptional points in open and closed gain–loss structures *Presented at PHHQPXI (Paris, 27–31 August)* to be submitted
[56] Lipkin H J, Meshkov N and Glick N 1965 *Nucl. Phys.* A **62** 188
Lipkin H J, Meshkov N and Glick N 1965 *Nucl. Phys.* A **62** 199
Lipkin H J, Meshkov N and Glick N 1965 *Nucl. Phys.* A **62** 211
[57] Heiss W D, Scholtz F G and Geyer H B 2005 *J. Phys. A: Math. Gen.* **38** 1843
[58] Leyvraz F and Heiss W D 2005 *Phys. Rev. Lett.* **95** 050402
[59] Cejnar P, Heinze S and Macek M 2007 *Phys. Rev. Lett.* **99** 100601
[60] Heiss W D and Sannino A L 1991 *Phys. Rev.* A **43** 4159
[61] Heiss W D 2010 *Eur. Phys. J.* D **60** 257
[62] Andrianov A A, Cannata F and Sokolov A V 2007 *Nucl. Phys.* B **773** 107
[63] Heiss W D 2008 *J. Phys. A: Math. Theor.* **41** 244010
[64] Demange G and Graefe E-M 2012 *J. Phys. A: Math. Theor.* **45** 025303
[65] Ryu J-W, Lee S-Y and Kim S W 2011 arXiv:1109.4216v1 [quant-ph]
[66] Seyranian A P and Mailybaev A A 2003 *Multiparameter Stability Theory with Mechanical Applications* (Singapore: World Scientific)
[67] Jones C A 1988 *Q. J. Mech. Appl. Math.* **41** 363
[68] Or A C 1991 *Q. J. Mech. Appl. Math.* **44** 559
[69] Busse F H and Or A C 1986 *J. Fluid Mech.* **166** 173
[70] Ring P and Schuck P 1980 *The Nuclear Many Body Problem* (New York: Springer)
[71] Heiss W D and Nazmitdinov R G 2002 *Phys. Rev.* C **65** 054304 and references therein
[72] Schucan T H and Weidenmüller H A 1972 *Ann. Phys., NY* **73** 108
[73] Bender C M and Wu T T 1968 *Phys. Rev. Lett.* **21** 406
[74] Bender C M and Wu T T 1969 *Phys. Rev.* **184** 1231
[75] Michel N, Nazarewicz W, Ploszajczak M and Vertse T 2009 *J. Phys. G: Nucl. Part. Phys.* **36** 013101
Michel N, Nazarewicz W, Okolowicz J and Ploszajczak M 2010 *J. Phys. G: Nucl. Part. Phys.* **37** 064042